$D = 4$, $N = 2$ Gauged Supergravity coupled to Vector-Tensor Multiplets

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Abstract

We construct the general four-dimensional $N = 2$ supergravity theory coupled to vector and vector-tensor multiplets only. Consistency of the construction requires the introduction of the vector fields dual to those sitting in the same supermultiplets as the antisymmetric tensors, as well as the scalar fields dual to the tensors themselves. Gauge symmetries also involving these additional fields guarantee the correct counting of the physical degrees of freedom.

1 Introduction

In the last decade, a considerable interest has been devoted to investigating the role of antisymmetric tensors (p-forms with $p > 1$) in four and five dimensional extended supergravity theories [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The study of Free Differential Algebras including gauge-coupled forms of higher order has an interest per se: It has been shown that for these theories the consistency of the 1-form sector does not imply the Jacobi identities, which indeed are no longer satisfied [13]. However it is still possible, via a field redefinition, to choose a setting where the Jacobi identities are satisfied, at the price of having a deformation of the gauge couplings in the Bianchi identities [14].

Moreover, antisymmetric tensors naturally appear in string compactifications and in supergravity theories with fluxes turned on. Focussing on four dimensional N-extended theories, the relevant tensors are given by 2-forms which, at the ungauged level, can always be dualized to real scalars. However, when gauge charges and masses are turned on, theories with tensor multiplets contain different couplings with respect to the ones where all tensors are dualized. In particular, the masses of the tensors appear as magnetic charges, so that the symplectic frame involved is in general quite different with respect to the standard theories. Some years ago, the N=2 theory coupled to hyper-tensor multiplets (obtainable from hypermultiplets upon dualizing some of the scalar fields into antisymmetric tensors [8]) was constructed [4, 5, 6]. It exhibited a symplectically invariant scalar potential in the sector related to the $SU(2)$ part of the R-symmetry. Eventually, the N=2 theory in five dimensions coupled to hyper-, vector-, and massive hypertensor multiplets was constructed [2, 10] and a symplectically covariant formulation of maximally extended theories in five and four dimensions, involving tensor fields, was given in [11, 12, 13], making use of the embedding tensor formalism [19, 20, 21]. Few years ago, some of the present authors studied on general grounds the Free Differential Algebra of forms of various degree based on the gauging of a general Lie algebra G [14] focussing on the Higgs mechanism through which the higher order forms get their mass. In this work, just as in [11], the "selfduality in odd dimensions" mechanism [22], on which [10] was based, originated from the gauge-fixing of a theory formulated in terms of gauge-coupled massless fields, via the Higgs mechanism. For the $N = 2$ theory in five dimensions, it was also pointed out in [14] that the Higgs mechanism has to be at work all *within* the same multiplet,

so that the massive tensor multiplets have to be BPS, i.e. short ones. In particular, the 2-form B_M acquires mass by "eating" the degrees of freedom of its Hodge-dual vector A^M.

A complete general formulation of N=2 supergravity in four dimensions coupled to vector-tensor multiplets was still missing. Some work in that direction was done in [1], where the coupling of $N = 2$ supergravity to a vector-tensor multiplet was first studied. Afterwards, in [23, 24] direct compactification of the five dimensional Lagrangian was given, and in [14] the supersymmetric Bianchi identities were solved up to three-fermions terms, giving the expression of the scalar potential and a set of constraints on the geometry of the σ-model.

The aim of this paper is to write the general Lagrangian of N=2, D=4 supergravity coupled to vector multiplets and vector-tensor multiplets. We find that, as for the five dimensional case, the tensor has to belong to a short representation of supersymmetry [1]. In this case, as will be discussed in the text, the Hodge duality acts non trivially: the tensor, B_M, becomes massive by eating the degrees of freedom of a vector, A^M, which is the Hodge-dual of the vector A_M in the multiplet; the vector A_M in turn gets mass by eating the degree of freedom of the scalar Hodge-dual to the tensor B_M. This in turn implies that the symplectic embedding is quite involved. A similar mechanism was described in [12, 13]. For this reason, the construction of the model requires to work in an enlarged field space where the Hodge-dual fields involved in the gauging are present together with the fields composing the multiplets. As a consequence, requiring supersymmetry and gauge invariance, we obtain a set of constraints on the fields and in particular the ones determining the geometry of the σ-model, which still need to be explicitly solved. The explicit analysis of the geometry of the manifold spanned by the scalars which survive the dualization into tensor fields is postponed to a forthcoming publication [25]. Moreover, the Lagrangian we get has a manifest symplectic invariance in the sector involving vector-tensor multiplets. It would be interesting to extend the symplectic invariance also to the vector multiplets sector. This would require, as explained in [7, 5, 6], the coupling of the theory also to hypermutiplets and hypertensor multiplets in a non abelian way. Alternatively, it could be found as a symplectically covariant gauging of the standard general matter coupled $N = 2$ supergravity, in the

[1] Note that this implies the presence of a central charge, in agreement with the results in [1].

spirit of the approach followed in [13] for the $N = 8$ theory. The extension of the construction to include hypermultplets and FI terms, is left to future investigation.

The paper is organized as follows:
Section 2 contains a description of the field content and of the peculiarities of the theory.
In section 3 we discuss the bosonic Free Differential Algebra underlying our theory, studying in particular the gauge structure and the symplectic embedding.
Sections 4 and 5 include the main results of our paper:
Section 4 contains the complete Lagrangian and supersymmetry transformation laws, while in Section 5 we comment on the results found and make some observations on the geometry of the embedding scalar manifold and on its relation with Special Geometry.
The Appendices contain technical and computational details:
Appendix A contains an explicit discussion of closure of the gauge algebra when embedded in the symplectic algebra.
In Appendix B we list the superspace Bianchi identities of the fields together with their rheonomic parametrizations.
In Appendix C we give the superspace rheonomic Lagrangian of the theory.
In Appendix D we collect all the constraints found on the fields, on the σ-model and on the gauging from solving the superspace Bianchi identities and the superspace field equations from the rheonomic Lagrangian.
Finally in Appendix E we study in detail the dualization procedure, showing that our kinetic Lagrangian can be obtained from that of standard $N = 2$ supergravity by dualization of some of the scalars in the vector multiplets into 2-form tensors.

2 The vector-tensor multiplet structure

Let us consider $N = 2$ supergravity in four dimensions with field content given by:

- The gravity multiplet:

$$(V^a_\mu, \psi_{A\mu}, \psi^A_\mu, A^0_\mu)\,,$$

(where a and μ denote space-time indices respectively flat and curved,

$A = 1, 2$ is an R-symmetry $SU(2)$ index and we have decomposed the gravitino in chiral (ψ_A) and anti-chiral (ψ^A) components),

- n_V vector multiplets:

$$(A_\mu, \lambda^A, \lambda_A, z)^r , \qquad r = 1, \cdots , n_V ,$$

where z^r are holomorphic coordinates on the special manifold $\mathcal{M}_V$ spanned by its scalar sector and λ^{rA} are chiral spin-1/2 fields (with complex conjugate antichiral component $\lambda^{\bar{r}}_A$),

- n_T vector-tensor multiplets:

$$\left(B_{\mu\nu}, A_\mu, \chi^A, \chi_A, P\right)_M , \qquad M = 1, \cdots , n_T ,$$

where P_M are real functions of the scalar fields spanning the real manifold $\mathcal{M}_T$, which can be chosen as "special" coordinates on $\mathcal{M}_T$. Here M is a representation index of the gauge group G. χ^A_M, χ_{MA} denote left- and right-hand components of Majorana spinors respectively.

This theory can be thought of as obtained from standard $N = 2$ supergravity coupled to vector multiplets by Hodge-dualization of, say, the imaginary part of a subset of the complex scalar fields parametrizing the special manifold. More precisely, starting from $n = n_V + n_T$ vector multiplets of $N = 2$ supergravity with scalar fields $z^i = (z^r, z^m)$ (with $m = 1, \cdots , n_T$, $i = 1, \cdots , n_V + n_T$), we shall find that the Hodge duality between tensor and scalar fields is given by:

$$H_{M|\mu\nu\rho} = -\frac{\mathrm{i}}{3}\sqrt{-g}\, \epsilon_{\mu\nu\rho\sigma}(p_{Mi}\nabla^\sigma z^i - \bar{p}_{M\bar{\imath}}\nabla^\sigma \bar{z}^{\bar{\imath}}) \tag{2.1}$$

$p_{Mi}(z, \bar{z}) = \nabla_i P_M$ converting coordinate indices into representation indices of the gauge group of the theory. This is an on-shell equation obtained from closure of Bianchi identities in superspace or, equivalently, from the equations of motion of the rheonomic Lagrangian. From the space-time point of view, this is equivalent to requiring closure of the supersymmetry algebra.

Note that (2.1) does not identify the Hodge dual of H_M with an exact form, say dY_M but, for the sake of simplicity, we shall find it convenient to refer to the degrees of freedom dual to the tensor fields as $Y^{(1)}_M = Y_{Mm} dy^m$.

To our knowledge, as anticipated in the introduction, the construction of such theory in full generality has not been achieved so far, even if important

steps in that direction have been given in ref. [24], where the four dimensional theory was obtained by dimensional reduction from five dimensions and the ensuing properties thoroughly analyzed. However, that approach does not catch the most general theory, being restricted to models with a five dimensional uplift. Finally, in [14] a relevant part of the construction has been carried out by some of the authors. In particular, in that paper we discussed the solution of Bianchi identities in superspace which, besides giving the general supersymmetry transformation laws and the constraints on the geometry of the relevant σ-models, also allows us to retrieve in principle the equations of motion of the theory.

In [14] we solved, up to three fermions, the Free Differential Algebra Bianchi identities in superspace following the so-called geometric (or rheonomic) approach of [26]. This will be the starting point of our development here. Since some of the notations and conventions have been changed here with respect to [14], for the benefit of the reader we expose in the present paper the main results found there. We recall that, in order for the Free Differential Algebra to close in superspace, it is necessary to include among the defining bosonic fields of the tensor multiplet sector, besides the vectors A_M and the tensors B_M, also their Hodge duals, that is the (auxiliary) vectors A^M, Hodge dual to the A_M, and the real scalars y^m, Hodge dual to the tensors B_M. The gauge group G is gauged by the vectors $A^\Lambda = (A^X, A^M)$, that is by the $A^X \equiv (A^0, A^r)$, A^0 being the graviphoton, together with the A^M.

Let us remark that, if we think of the theory as dualization of ordinary $N = 2$ supergravity coupled to vector multiplets, the theory with tensor multiplets is in a rotated symplectic frame. Depending on whether the theory is thought of as constructed directly from vector-tensor multiplets or as dualization of standard $N = 2$, the interpretation of the vectors A^M, A_M is different. Indeed, in the former case the A_M are the physical fields, to be considered as electric, while the gauge group includes the magnetic vectors A^M. On the other hand, in the second interpretation the A^M are electric and the A_M are magnetic fields. In writing the Lagrangian, we will consider A^M as the propagating gauge fields. It will be useful to adopt a collective gauge-vector index $\Lambda = (X, M) = 0, 1, \cdots, n_V + n_T$ (with $X = 0, 1, \cdots, n_V$) running over the corresponding vectors of the theory. In the study of the supersymmetric Free Differential Algebra of the theory, we shall let all the vectors A^Λ be the gauge vectors of a non abelian algebra G and the tensors

B_M be in a representation of it [2].

In the interacting theory, the Higgs mechanism takes place so that the vectors A^M provide the degrees of freedom giving mass to the tensors B_M. In this way the gauge algebra is broken to a particular contraction G_0 ($\dim G_0 = n_V + 1$) spanned by the vectors (A^0, A^r). On the other hand, the gauge vectors A_M undergo a dual Higgs mechanism, since they take mass by eating the degrees of freedom of the dualized scalars y^m, and they will appear in the supercurvature of the tensor field-strengths H_M. As already remarked, if we did not include all the fields together with their Hodge duals, inconsistencies would show up in the superspace Bianchi identities. Note that our approach has been to introduce the dual fields as auxiliary fields, letting closure of the free differential algebra and the Lagrangian equations of motion determine them in terms of the physical fields as their Hodge duals. Since the fields y^m have to be included for a correct description of the dynamics of the theory, it is convenient to adopt a complex notation also for the vector-tensor sector and work with holomorphic coordinates $z^m \equiv z^m(P_M, y^m)$ together with their complex conjugates $\overline{z}^{\overline{m}}$ (with $m, \overline{m} = 1, \cdots, n_T$). Using this notation, it is natural to introduce a collective holomorphic world-index $i = (r, m) = 1, \cdots, n_V + n_T$ on the $2(n_V + n_T)$-dimensional embedding manifold $\mathcal{M}_{(emb)}$, in parallel to what has been done for gauge indices. This notation is quite natural from the point of view of dualization of the standard supergravity theory, where z^m are part of the Kähler coordinates z^i. According to it, we will extend the set of spinors λ^r to λ^i, including among them the spinors χ_M, such that $\chi^A_M = p_{Mi}\lambda^{iA}$, $\chi_{MA} = \bar{p}_{M\bar{\imath}}\lambda^{\bar{\imath}}_A$. Using the collective index formalism the theory will look quite like the standard $N = 2$ supergravity coupled to vector multiplets only, and this explains, as we will see in the following, that most of the results coming from Bianchi Identities will look formally like those of the standard $N = 2$ supergravity, or a suitable extension of it. Since then the Free Differential Algebra involves both the antisymmetric tensors B_M and the degrees of freedom y^m, we expect that the closure of the superspace Bianchi identities should imply the duality relation between them. In fact this is what happens, see eq. (2.1), implying that the dualization relation is valid only on-shell. As a consequence, the on-shell geometry will look rather different from its off-shell counterpart. In particular, in the absence of a

[2] As discussed in [14], starting from a general algebra G, with the constraint on the structure constants $f_{\Lambda M}{}^X = 0$, we can always retrieve, by a suitable redefinition of the 2-forms B_M, the case $G = G_0 \ltimes \mathbb{R}^{n_T}$, where G_0 is in general a contraction of G gauged by the vectors A^X.

factorization of the two σ-models $\mathcal{M}_T$ and $\mathcal{M}_V$, the off-shell Kähler–Hodge structure is completely destroyed since the metric is not even hermitian.

We finally note that, exactly like in the five dimensional case, the massive vector-tensor multiplets of the $N = 2$ four dimensional theory are short, BPS multiplets. This is in a contrast with what happens for the scalar-tensor multiplets, where the tensor field is Hodge-dual to a scalar in the hypermultiplet sector [6]. In that case, the multiplet becomes massive by introducing an appropriate coupling to a vector multiplet. In our case, instead, the degrees of freedom corresponding to A^M and y^m do not have spinor partners, but act as bosonic Lagrange multipliers in the theory. Being BPS multiplets, they are therefore charged and this in turn requires for CPT invariance that the vector-tensor multiplet sector always includes an even number of tensor fields.

3 The structure of the bosonic Free Differential Algebra

Let us summarize here the main properties of the bosonic Free Differential Algebra underlying the supergravity theory, found in [14]. [3] It reads:

$$\left\{\begin{array}{rcl} F^\Lambda &=& dA^\Lambda + \frac{1}{2} f_{\Sigma\Gamma}{}^\Lambda A^\Sigma \wedge A^\Gamma + m^{\Lambda M} B_M \\ F_M &=& dA_M + \hat{T}_{\Lambda M}{}^N A^\Lambda \wedge A_N \\ H_M &=& dB_M + T_{\Lambda M}{}^N A^\Lambda \wedge B_N + \Big(d_{\Lambda\Sigma M} A^\Sigma + \hat{T}_{\Lambda M}{}^N A_N \Big) \wedge F^\Lambda \end{array}\right. \tag{3.1}$$

and it closes the Bianchi identities

$$\left\{\begin{array}{rcl} \nabla F^\Lambda &=& m^{\Lambda M} H_M \\ \nabla F_M &=& 0 \\ \nabla H_M &=& \Big(d_{\Lambda\Sigma M} F^\Sigma + \hat{T}_{\Lambda M}{}^N F_N \Big) \wedge F^\Lambda \end{array}\right. \tag{3.2}$$

where the covariant derivatives are defined as follows:

$$\nabla F^\Lambda \equiv dF^\Lambda + \hat{f}_{\Lambda\Sigma}{}^\Gamma A^\Sigma \wedge F^\Gamma + m^{\Lambda M} \hat{T}_{\Sigma M}{}^N A_N \wedge F^\Sigma \tag{3.3}$$

$$\nabla F_M \equiv dF_M + \hat{T}_{\Lambda M}{}^N \left(A^\Lambda \wedge F_N - A_N \wedge F^\Lambda \right) \tag{3.4}$$

$$\nabla H_M \equiv dH_M + \hat{T}_{\Lambda M}{}^N A^\Lambda \wedge H_N \tag{3.5}$$

[3]With an abuse of notation, we will call here with the same name H_M, F_M and F^Λ the bosonic field-strengths associated to the corresponding forms on superspace.

provided the following constraints are satisfied:

$$
\begin{aligned}
f_{[\Sigma\Gamma}{}^{\Delta} f_{\Lambda]\Delta}{}^{\Pi} &= 0 && (3.6)\\
T_{[\Lambda M}{}^{P} T_{\Sigma]P}{}^{N} &= \frac{1}{2} f_{\Lambda\Sigma}{}^{\Gamma} T_{\Gamma M}{}^{N} && (3.7)\\
T_{\Lambda M}{}^{N} &= d_{\Lambda\Sigma M} m^{\Sigma N} && (3.8)\\
m^{\Lambda(P} T_{\Lambda M}{}^{N)} &= 0 && (3.9)\\
f_{\Sigma\Gamma}{}^{\Lambda} m^{\Gamma M} &= m^{\Lambda N} T_{\Sigma N}{}^{M} && (3.10)\\
\hat{T}_{[\Lambda|M}{}^{N} d_{\Gamma|\Sigma]N} &- f_{[\Lambda|\Gamma}{}^{\Delta} d_{\Delta|\Sigma]M} - \frac{1}{2} f_{\Lambda\Sigma}{}^{\Delta} d_{\Gamma\Delta M} = 0 && (3.11)
\end{aligned}
$$

In the relations above we used the definition:

$$
\begin{aligned}
\hat{T}_{\Lambda M}{}^{N} &\equiv T_{\Lambda M}{}^{N} + d_{\Sigma\Lambda M} m^{\Sigma N} = 2 d_{(\Lambda\Sigma)M} m^{\Sigma N} && (3.12)\\
\hat{f}_{\Sigma\Gamma}{}^{\Lambda} &\equiv f_{\Sigma\Gamma}{}^{\Lambda} + d_{\Gamma\Sigma M} m^{\Lambda M} && (3.13)
\end{aligned}
$$

From (3.6)-(3.11) and (3.12), (3.13) we derive the further, useful relations (see [14]):

$$
\begin{aligned}
\hat{f}_{\Lambda\Gamma}{}^{\Delta} \hat{f}_{\Sigma\Delta}{}^{\Pi} - \hat{f}_{\Sigma\Gamma}{}^{\Delta} \hat{f}_{\Lambda\Delta}{}^{\Pi} &= -\hat{f}_{\Lambda\Sigma}{}^{\Delta} \hat{f}_{\Delta\Gamma}{}^{\Pi} && (3.14)\\
\hat{T}_{[\Lambda M}{}^{P} \hat{T}_{\Sigma]P}{}^{N} &= \frac{1}{2} f_{\Lambda\Sigma}{}^{\Gamma} \hat{T}_{\Gamma M}{}^{N} && (3.15)\\
m^{M(P} \hat{T}_{\Sigma M}{}^{N)} &= 0 && (3.16)\\
\hat{f}_{\Sigma\Gamma}{}^{\Lambda} m^{\Gamma M} &= m^{\Lambda N} \hat{T}_{\Sigma N}{}^{M} && (3.17)\\
\hat{f}_{\Sigma\Gamma}{}^{\Lambda} m^{\Sigma M} &= 0 && (3.18)\\
m^{\Lambda P} \hat{T}_{\Lambda M}{}^{N} &= 0 && (3.19)
\end{aligned}
$$

Let us observe, following [14], that the Free Differential Algebra written above contains, in the definition of the field strengths, gauge couplings different from the ones in the Bianchi identities (non-"hatted" versus "hatted" couplings), that is the couplings in the covariant derivatives and Bianchi identities are deformed with respect to the defining ones. This is an unavoidable peculiarity of our request (3.6) of closure of the gauge algebra Jacobi identities, a consistency condition that must be satisfied by the gauge algebra G if we require it to be an electric algebra. However, as shown in [14], via the field redefinition

$$B_M \to B_M + d_{\Lambda\Sigma M} A^{\Lambda} \wedge A^{\Sigma}$$

the Free Differential Algebra can be put in a form where only the hatted couplings appear. In this new, equivalent setting, however, closure of the gauge algebra underlying the Free Differential Algebra is not manifest, since the Jacobi identities are not satisfied. This is the setting generally used in [13]. Note that in [13], where the Free Differential Algebra was studied in the embedding tensor framework, the closure of the algebra was however guaranteed by the fact that the embedding tensor should satisfy a set of quadratic constraints, that are precisely the same as (3.6)-(3.11).

3.1 Gauge invariance properties and symplectic embedding

Let us study in more detail the gauge structure of the Free Differential Algebra (3.1).

3.1.1 2-form gauge transformation

Eq. (3.1) is invariant under the 2-form gauge transformation with 1-form parameter Λ_M:

$$\left\{\begin{array}{rcl} \delta B_M &=& d\Lambda_M + T_{\Lambda M}{}^N A^\Lambda \wedge \Lambda_N \equiv D\Lambda_M \\ \delta A^\Lambda &=& -m^{\Lambda M}\Lambda_M \\ \delta A_M &=& 0 \end{array}\right. , \tag{3.20}$$

under which

$$\left\{\begin{array}{rcl} \delta H_M &=& 0 \\ \delta F^\Lambda &=& 0 \\ \delta F_M &=& 0 \end{array}\right. . \tag{3.21}$$

However, we still have the freedom to redefine $B_M \to B_M + k_{\Lambda\Sigma M} A^\Lambda \wedge A^\Sigma$ for a generic constant tensor $k_{\Lambda\Sigma M}$. We can exploit this freedom to fix the 1-form gauge $\bar\Lambda_M$ such that:

$$\left\{\begin{array}{rcl} A^\Lambda &\to& A'^\Lambda = \delta^\Lambda_X A^X - m^{\Lambda M}\bar\Lambda_M \\ A_M &\to& A'_M = A_M \\ B_M &\to& B'_M = \bar B_M + d\bar\Lambda_M + T_{XM}{}^N A^X \bar\Lambda_N + \\ && -\frac{1}{2} d_{XYM} A^X \wedge A^Y - \frac{1}{2} d_{\Lambda\Sigma M} m^{\Lambda N} m^{\Sigma P} \bar\Lambda_N \wedge \bar\Lambda_P \end{array}\right. ,$$

the free differential agebra turns out to be written only in terms of physical massive fields:

$$\left\{\begin{array}{rcl} F'^{\Lambda} &=& \delta^{\Lambda}_{X} F^{X} + m^{\Lambda M}\bar{B}_{M} \\ F'_{M} &=& F_{M} \\ H'_{M} &=& d\bar{B}_{M} + \hat{T}_{XM}{}^{N}\left(A^{X}\bar{B}_{N} + A_{N}F^{X}\right) + d_{(XY)M}A^{X}F^{Y} + \\ && + \left(f_{XY}{}^{W}d_{[ZW]M} + T_{XM}{}^{N}d_{YZN}\right)A^{X}\wedge A^{Y}\wedge A^{Z} \end{array}\right. ,$$

3.1.2 1-form gauge transformations and symplectic embedding

The Free Differential Algebra is also covariant under the 1-form gauge transformation with parameters ϵ^{Λ}, ϵ_{M}:

$$\left\{\begin{array}{rcl} \delta A^{\Lambda} &=& d\epsilon^{\Lambda} + f_{\Sigma\Gamma}{}^{\Lambda}A^{\Sigma}\epsilon^{\Gamma} \\ \delta A_{M} &=& d\epsilon_{M} + \hat{T}_{\Lambda M}{}^{N}\left(A^{\Lambda}\epsilon_{N} - A_{N}\epsilon^{\Lambda}\right) \\ \delta B_{M} &=& -T_{\Lambda M}{}^{N}\epsilon^{\Lambda}B_{N} - \left(d_{\Lambda\Sigma M}\epsilon^{\Sigma} + \hat{T}_{\Lambda M}{}^{N}\epsilon_{N}\right)F^{\Lambda} \end{array}\right. , \tag{3.22}$$

under which:

$$\left\{\begin{array}{rcl} \delta F^{\Lambda} &=& -\hat{f}_{\Gamma\Sigma}{}^{\Lambda}F^{\Sigma}\epsilon^{\Gamma} - m^{\Lambda N}\hat{T}_{\Sigma N}{}^{M}F^{\Sigma}\epsilon_{M} \\ \delta F_{M} &=& \hat{T}_{\Lambda M}{}^{N}\left(F^{\Lambda}\epsilon_{N} - F_{N}\epsilon^{\Lambda}\right) \\ \delta H_{M} &=& -\hat{T}_{\Lambda M}{}^{N}\epsilon^{\Lambda}H_{N} \end{array}\right. . \tag{3.23}$$

Let us emphasize that, as explained in Section 2, the theory contains electric gauge vectors $A^{\Lambda} = (A^{X}, A^{M})$ together with magnetic ones A_{M}, since all of them are needed to implement the Higgs mechanism giving mass to the vector-tensor multiplet. In particular, as we will see in the following, the equations of motion of the B-fields identify, on-shell, the field-strengths $F_{M} = \mathcal{N}_{M\Lambda}F^{+\Lambda} + \overline{\mathcal{N}}_{M\Lambda}F^{-\Lambda}$ (see eq. (5.3)) with the magnetic field-strengths $\mathcal{G}_{M} \equiv -\frac{1}{2}{}^{*}\frac{\delta}{\delta F^{M}}(\mathcal{L}_{k} + \mathcal{L}_{Pauli})$, where $\mathcal{L}_{k}$ and $\mathcal{L}_{Pauli}$ are the kinetic and Pauli Lagrangians respectively. Then, we can write a symplectic vector of electric and magnetic field strengths as:

$$\mathcal{F}^{\alpha} = \left(F^{\Lambda}, \mathcal{G}_{X}, F_{M}\right) ,$$

with $\alpha = 1, \cdots, 2(1 + n_{V} + n_{T})$ being a symplectic index running over all the electric and magnetic fields. $\mathcal{G}_{X} \equiv -\frac{1}{2}{}^{*}\frac{\delta}{\delta F^{X}}(\mathcal{L}_{k} + \mathcal{L}_{Pauli})$ is the magnetic field-strength dual to F^{X}. Then, the gauge variations of the field-strenghts, eq. (3.23), can be written as:

$$\delta_{\epsilon}\mathcal{F}^{\alpha} = -\mathcal{F}^{\gamma}\left(\mathcal{T}_{\beta}\right)_{\gamma}{}^{\alpha}\epsilon^{\beta} \tag{3.24}$$

where $\mathcal{T}_\alpha$ are the gauge algebra generators embedded in the symplectic group. They can be written in block form as

$$(\mathcal{T}_\alpha)_\beta{}^\gamma = \begin{pmatrix} (\mathcal{T}_\alpha)_\Sigma{}^\Gamma & (\mathcal{T}_\alpha)_{\Sigma\Gamma} \\ (\mathcal{T}_\alpha)^{\Sigma\Gamma} & (\mathcal{T}_\alpha)^\Sigma{}_\Gamma \end{pmatrix} \tag{3.25}$$

where, for $\mathcal{T}$ to be symplectic,

$$(\mathcal{T}_\alpha)^\Sigma{}_\Gamma = -(\mathcal{T}_\alpha)_\Sigma{}^\Gamma, \quad (\mathcal{T}_\alpha)_{\Sigma\Gamma} = (\mathcal{T}_\alpha)_{\Gamma\Sigma}, \quad (\mathcal{T}_\alpha)^{\Sigma\Gamma} = (\mathcal{T}_\alpha)^{\Gamma\Sigma} \tag{3.26}$$

Condition (3.26), together with (3.23), allows to completely determine the $\mathcal{T}_\alpha$:

$$\begin{aligned}
(\mathcal{T}_\Lambda)_\beta{}^\gamma &= \begin{pmatrix} \hat{f}_{\Lambda\Sigma}{}^\Gamma & (\mathcal{T}_\Lambda)_{XY}\delta^X_{(\Sigma}\delta^Y_{\Gamma)} \\ 0 & -\hat{f}_{\Lambda\Gamma}{}^\Sigma \end{pmatrix} && (3.27)\\
(\mathcal{T}^P)_\beta{}^\gamma &= \begin{pmatrix} m^{\Gamma N}\hat{T}_{\Sigma N}{}^P & -2\delta^N_{(\Sigma}\hat{T}_{\Gamma)N}{}^P + (\mathbf{T}^P)_{XY}\delta^X_{(\Lambda}\delta^Y_{\Sigma)} \\ 0 & -m^{\Sigma N}\hat{T}_{\Gamma N}{}^P \end{pmatrix} && (3.28)\\
(\mathcal{T}^X)_\beta{}^\gamma &= 0 && (3.29)
\end{aligned}$$

Note that the tensor $(\mathcal{T}_\Lambda)_{XY}$ is not relevant in the vector-tensor sector of the theory, as it is not coupled to fields in the tensor multiplets, however it corresponds to a possible deformation of the gauging in the vector multiplet directions, for gaugings having a non-homogeneous action on the vector-kinetic matrix [27]. The symplectic embedding above allows to predict the form of the gauge transformation of the magnetic field-strengths $\mathcal{G}_X$:

$$\begin{aligned}
\delta\mathcal{G}_X &= -\left(\mathcal{T}_{\Lambda XY}\epsilon^\Lambda + (\mathcal{T}^P)_{XY}\epsilon_P\right) F^Y + \hat{T}_{XN}{}^P\,\epsilon_P\,F^N + \\
&\quad + f_{\Lambda X}{}^Y\,\epsilon_\Lambda\,\mathcal{G}_Y + \left(\hat{f}_{\Lambda X}{}^N\,\epsilon_\Lambda + \hat{T}_{XM}{}^P\,m^{NM}\,\epsilon_P\right) F_N\,. && (3.30)
\end{aligned}$$

The generators $\mathcal{T}_\alpha$ have to satisfy a set of relations corresponding to a no-anomaly condition [28], which is easily expressed in the embedding tensor formalism and reads:

$$(\mathcal{T}_{(\alpha})_\beta{}^\delta\,\mathbb{C}_{\gamma)\delta} = 0\,. \tag{3.31}$$

It corresponds to the following relations:

$$\begin{aligned}
(\mathcal{T}_{(\Lambda})_{\Sigma\Gamma)} &= 0 && (3.32)\\
(\mathcal{T}^\Gamma)_{\Lambda\Sigma} &= 2(\mathcal{T}_{(\Lambda})_{\Sigma)}{}^\Gamma && (3.33)\\
(\mathcal{T}_\Gamma)^{\Lambda\Sigma} &= -2(\mathcal{T}^{(\Lambda})_\Gamma{}^{\Sigma)} && (3.34)\\
(\mathcal{T}^{(\Lambda})^{\Sigma\Gamma)} &= 0 && (3.35)
\end{aligned}$$

In particular, eq. (3.32) implies

$$(\mathcal{T}_M)_{XY} = 0\,, \tag{3.36}$$

eq. (3.33), using (3.12), implies

$$\begin{aligned} m^{\Gamma M} d_{(\Lambda\Sigma\Gamma)} &= \frac{1}{6}(\mathbf{T}^M)_{XY}\delta^X_{(\Lambda}\delta^Y_{\Sigma)}\,, && (3.37)\\ m^{XM}\, d_{(\Lambda N)M} &= 0\,. && (3.38) \end{aligned}$$

Eq. (3.37) in turns implies, by multiplication with $m^{\Sigma N}$

$$m^{M(P}\hat{T}_{\Sigma M}{}^{N)} \;=\; 0\,, \tag{3.39}$$

while eq. (3.38) is trivially satisfied if we choose, as we will do throughout the paper, a basis where

$$m^{XP} = 0 \quad , \quad \det[m^{MP}] \neq 0\,. \tag{3.40}$$

Eq. (3.34), using (3.9) and (3.39), implies

$$(\mathcal{T}_\Gamma)^{\Lambda\Sigma} \;=\; 0\,, \tag{3.41}$$

while eq. (3.35) is trivially satisfied. Let us finally observe that the following relation holds: $(\mathcal{T}_\Lambda)_\alpha{}^\beta m^{\Lambda M} = 0$ as is easily checked using (3.6) - (3.11). In the basis (3.40), it corresponds to the statement $\mathcal{T}_M = 0$.

The $\mathcal{T}_\alpha$, subject to (3.6) - (3.11) together with (3.32) - (3.35), close the algebra in the symplectic representation:

$$[\mathcal{T}_\alpha, \mathcal{T}_\beta] = -\mathcal{T}_{\alpha\beta}{}^\gamma\, \mathcal{T}_\gamma\,. \tag{3.42}$$

This will be shown in Appendix A.

Note that the eqs. (3.32) - (3.35) are necessary conditions to have gauge and supersymmetry invariance of the $N = 2$ Lagrangian including topological terms of generalized Chern–Simons type. The symplectic embedding built up above is crucial to show this property. Indeed, let us recall that in the gauged theory, according to [29], the gauge group has to be embedded in the symplectic group, as we found above (see (3.27), (3.28)). We generally have, for an infinitesimal electric symplectic rotation $S = \mathbb{1} - s$ with $s = \begin{pmatrix} a & c \\ 0 & -a^t \end{pmatrix}$ of the (self-dual part of the) field strength $(\mathcal{F}^-)^\alpha$:

$$(\mathcal{F}'^-)^\beta = (\mathcal{F}^-)^\alpha S_\alpha{}^\beta \tag{3.43}$$

implying, recalling that $\mathcal{G}^-_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma}(F^-)^\Sigma$

$$\delta\mathcal{N} = -c + a^t\mathcal{N} + \mathcal{N}a\,. \tag{3.44}$$

In extended supergravity the gauge algebra has to be embedded in the symplectic algebra. In particular, for an electric theory the infinitesimal gauge transformations are given by a matrix s where [30]:

$$s_\alpha{}^\beta = \epsilon^\Lambda\left(\mathcal{T}_\Lambda\right)_\alpha{}^\beta \tag{3.45}$$

For theories including magnetic vectors, the above relation generalizes to [12, 13]

$$s_\alpha{}^\beta = \epsilon^\Lambda\left(\mathcal{T}_\Lambda\right)_\alpha{}^\beta + \epsilon_\Lambda\left(\mathcal{T}^\Lambda\right)_\alpha{}^\beta \tag{3.46}$$

This implies that, when the constant matrix c is different from zero, the supersymmetric and gauge invariant Lagrangian must include a generalized Chern–Simons topological term [27]. In particular, in our case we have

$$c_{\Lambda\Sigma} = \epsilon^\Gamma(\mathcal{T}_\Gamma)_{\Lambda\Sigma} + \epsilon_M(\mathcal{T}^M)_{\Lambda\Sigma} \tag{3.47}$$

It is worth noticing that the gauge invariance of the Lagrangian requires (3.32) - (3.35), together with the further conditions (antisymmetrization in $\Lambda_1\Lambda_2\Lambda_3\Lambda_4$ is understood in (3.49)):

$$f_{[\Lambda\Sigma}{}^\Omega(\mathcal{T}_{\Gamma]})_{\Delta\Omega} = f_{[\Lambda\Sigma}{}^\Omega(\mathcal{T}_\Gamma)_{\Delta]\Omega} \tag{3.48}$$

$$\begin{aligned} f_{\Lambda_1\Lambda_2}{}^\Theta\Big[&f_{\Theta\Delta}{}^\Omega(\mathcal{T}_{\Lambda_4})_{\Lambda_3\Omega} - f_{\Lambda_3\Lambda_4}{}^\Omega(\mathcal{T}_\Delta)_{\Theta\Omega} + f_{\Delta\Lambda_4}{}^\Omega(\mathcal{T}_{\Lambda_3})_{\Theta\Omega} - \\ &+f_{\Delta\Lambda_4}{}^\Omega(\mathcal{T}_\Omega)_{\Lambda_3\Theta}\Big] = 0 \end{aligned} \tag{3.49}$$

These relations do not involve the tensor sector, since they only include electric couplings. It is anyway interesting to analyze them in more detail, in the context of the gaugings introduced in [27]. In fact, eq. (3.48) coincides with the condition (3.18) in [27]. As far as eq. (3.49) is concerned, actually it has a geometric meaning since it corresponds to a cohomological statement. Indeed, let us consider the tensor $t_{\Lambda\Sigma\Gamma\Delta} \equiv f_{[\Lambda\Sigma}{}^\Omega(\mathcal{T}_{\Gamma]})_{\Delta\Omega} = -\frac{1}{2}t_{[\Lambda\Sigma\Gamma\Delta]}$. Eq. (3.49) can be rewritten, using (3.48), as:

$$3t_{\Sigma\Gamma[\Lambda_1\Lambda_2}f_{\Lambda_3\Lambda_4]}{}^\Gamma + 2f_{\Sigma[\Lambda_1}{}^\Gamma t_{\Lambda_2\Lambda_3\Lambda_4]\Gamma} = 0 \tag{3.50}$$

This equation has a simple interpretation in terms of the Chevalley–Eilenberg Lie algebra cohomology of the gauge group G [31]. Indeed, the free differential

algebra (3.1) is constructed starting from the algebra of the gauge group, which in dual form is expressed by the Cartan–Maurer equation [26]:

$$dA^\Lambda + \frac{1}{2} f_{\Sigma\Gamma}{}^\Lambda A^\Sigma \wedge A^\Gamma = 0\,. \tag{3.51}$$

We recall that for a generic p-form in a given representation $D(\mathbf{T}_\Lambda)$ of G labeled by the index K

$$\omega_K^{(p)} = \omega_{K|\Lambda_1\cdots\Lambda_p}^{(p)} A^{\Lambda_1} \wedge \cdots \wedge A^{\Lambda_p}$$

the condition for $\omega_K^{(p)}$ to be a representative of a cohomology class $H^{(p)}$ of the Lie algebra is

$$\partial\omega_K^{(p)} = 0$$

where

$$\partial\omega_K^{(p)} \equiv \nabla\omega_K^{(p)} = -\frac{1}{2} f_{\Sigma\Gamma}{}^\Lambda A^\Sigma \wedge A^\Gamma \wedge (i_\Lambda \omega_K^{(p)}) + D(\mathbf{T}_\Lambda)_K{}^L \wedge \omega_L^{(p)} \tag{3.52}$$

and i_Λ denotes contraction along the generator $\mathbf{T}_\Lambda$. Then if we consider $t_{\Lambda\Sigma\Gamma\Delta}$ as the component of a 3-form in the adjoint representation of G:

$$t_\Lambda \equiv t_{\Lambda\Sigma\Gamma\Delta} A^\Sigma \wedge A^\Gamma \wedge A^\Delta \tag{3.53}$$

then eq. (3.50) is just the condition that t_Λ lies in the cohomology class $H^{(3)}$ of the Chevalley–Eilenberg cohomology of the $\{A^\Lambda\}$ [4]. Notice moreover that also eq. (3.11) has a Lie algebra cohomology interpretation: indeed, introducing the 1-form $\Phi_{\Lambda M} \equiv d_{\Lambda\Sigma M} A^\Sigma$, it can be easily verified that (3.11) corresponds the condition for $\Phi_{\Lambda M}$ to lie in the Chevalley–Eilenberg cohomology class $H^{(1)}$.

4 The $N = 2$ theory of supergravity coupled to vector and vector-tensor multiplets

4.1 Definition of Superspace Curvatures

According to the geometric approach, we define the Free Differential Algebra of our theory as follows:

[4] This is a general statement. Note that it extends eq. (3.17) of [27] also to theories where a prepotential $F(X)$ does not exist.

- We start with the Free Differential Algebra of pure supergravity:

$$\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b \tag{4.1}$$
$$\mathcal{T}^a = \mathrm{d}V^a - \omega^a{}_b V^b - \mathrm{i}\overline{\psi}_A \gamma^a \psi^A \tag{4.2}$$
$$\rho_A = \mathrm{d}\psi_A - \frac{1}{4}\omega_{ab}\gamma^{ab}\psi_A + \frac{\mathrm{i}}{2}\mathcal{Q}\psi_A \tag{4.3}$$
$$\rho^A = \mathrm{d}\psi^A - \frac{1}{4}\omega_{ab}\gamma^{ab}\psi^A - \frac{\mathrm{i}}{2}\mathcal{Q}\psi^A\,. \tag{4.4}$$

where with $\mathcal{Q}$ we denote a gauged $U(1)$ connection, which is the remnant of the gauged $U(1)$-Kähler composite connection of special geometry. We recall that in the these definitions the spin connection $\omega^a{}_b$ and the bosonic and fermionic component of the supervielbein V^a, ψ_A, ψ^A, as well as $\mathcal{Q}$, are superspace 1-forms , the left-hand sides definining the corresponding superspace curvatures.[5]

- To complete the superspace Free Differential Algebra, the bosonic spacetime fields and curvatures introduced in Section 3 for the vector and tensor field-strengths are suitably generalized to their superspace extension as follows:

$$F^\Lambda = \mathrm{d}A^\Lambda + \frac{1}{2}f_{\Sigma\Gamma}{}^\Lambda A^\Sigma A^\Gamma + m^{\Lambda M}B_M + L^\Lambda \overline{\psi}^A \psi^B \epsilon_{AB} + \overline{L}^\Lambda \overline{\psi}_A \psi_B \epsilon^{AB} \tag{4.5}$$
$$F_M = \mathrm{d}A_M + \hat{T}_{\Lambda M}{}^N A^\Lambda A_N + L_M \overline{\psi}^A \psi^B \epsilon_{AB} + \overline{L}_M \overline{\psi}_A \psi_B \epsilon^{AB} \tag{4.6}$$
$$\begin{aligned} H_M &= \mathrm{d}B_M + T_{\Lambda M}{}^N A^\Lambda B_N + 2\mathrm{i}P_M \overline{\psi}_A \gamma_a \psi^A V^a + \\ &\quad + \left(d_{\Lambda\Sigma M}A^\Sigma + \hat{T}_{\Lambda M}{}^N A_N\right)\left(F^\Lambda - L^\Lambda \overline{\psi}^A \psi^B \epsilon_{AB} - \overline{L}^\Lambda \overline{\psi}_A \psi_B \epsilon^{AB}\right) \end{aligned} \tag{4.7}$$

Here P_M is a real section on the σ-model, while L^Λ, $\overline{L}^\Lambda$, L_M and $\overline{L}_M$ are complex sections on the σ-model, analogous to the covariantly holomorphic sections of special geometry.

- Finally, the Free Differential Algebra is enlarged to include the 1-form gauged field-strengths for the 0-form complex scalars z^i and 0-forms

[5] For the definition of the gauged $U(1)$-Kähler composite connection of special geometry in terms of the ungauged one and for all the notation concerning special geometry, we refer the reader to the standard $N=2$, $D=4$ supergravity of ref. [32].

spin 1/2 spinors $\lambda^{iA}, \lambda^{\bar{\imath}}_{A}$ belonging to the $N = 2$ vector multiplet and tensor multiplet representations of supersymmetry:

$$\nabla z^i = \mathrm{d}z^i + k^i_\Lambda A^\Lambda - k^{iM} A_M \tag{4.8}$$

$$\nabla \lambda^{iA} = \mathrm{d}\lambda^{iA} - \frac{1}{4}\omega_{ab}\gamma^{ab}\lambda^{iA} - \frac{\mathrm{i}}{2}\mathcal{Q}\lambda^{iA} + \Gamma^i{}_j \lambda^{jA} \tag{4.9}$$

where $\Gamma^i{}_j = \Gamma^i{}_{jk}\nabla z^k$ is the gauged Kählerian Levi–Civita connection $(1,0)$-form on the embedding σ-model $\mathcal{M}_{(emb)}$. k^i_Λ are the complex Killing vectors in the adjoint representation of the algebra G while k^{iM} are Killing vectors in the appropriate representation of G_0 (the invariant subgroup of G or, more generally, its contraction). This choice complies, in our redundant formulation, to the requirement that the vectors A_M undergo the Higgs mechanism by eating the real degrees of freedom Y^M dual to B_M. This will prove to be consistent with the solution of the superspace Bianchi identities.

The construction of the theory, namely the supersymmetric Lagrangian, its transformation laws and the constraints on the σ-model, is obtained by working out the constraints obtained from the superspace Bianchi identities and/or the superspace equations of motion of the rheonomic Lagrangian thought of as a 4-form embedded in superspace. A short derivation and a summary of the results are given in Appendices B, C and D. Restricting the rheonomic Lagrangian to the physical space-time we arrive at the following *space-time Lagrangian*:

$$\mathcal{S} = \int \sqrt{-g}\, d^4x \;[\mathcal{L}_k + \mathcal{L}_{Pauli} + \mathcal{L}_{gauge} + \mathcal{L}_{4f}] \tag{4.10}$$

where

$$\begin{aligned}\mathcal{L}_k = &-\frac{1}{2}\mathcal{R} + \left(\overline{\psi}^A_\mu \gamma_\sigma \rho_{A|\nu\rho} - \overline{\psi}_{A\mu}\gamma_\sigma \rho^A_{\nu\rho}\right)\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} + \mathrm{i}\left(\bar{\mathcal{N}}_{\Lambda\Sigma}\tilde{\mathcal{F}}^{-\Lambda}_{\mu\nu}\tilde{\mathcal{F}}^{-\Sigma|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\tilde{\mathcal{F}}^{+\Lambda}_{\mu\nu}\tilde{\mathcal{F}}^{+\Sigma|\mu\nu}\right) + \\ &+ \frac{1}{16}\mathcal{M}^{MN}Y_{M|\mu}Y^\mu_N - \frac{1}{8}\mathcal{M}^{MN}Y_{M|\mu}\tilde{H}_{N|\nu\rho\sigma}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} + \\ &+ \frac{\mathrm{i}}{4}\mathcal{M}^{MN}\tilde{H}_{M|\mu\nu\rho}\left(p_{Ni}\tilde{Z}^i_\sigma - \bar{p}_{N\bar{\imath}}\tilde{\bar{Z}}^{\bar{\imath}}_\sigma\right)\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} + \\ &+ \frac{1}{2}G_{ij}\tilde{Z}^i_\mu \tilde{Z}^{j|\mu} + G_{i\bar{\jmath}}\tilde{Z}^i_\mu \tilde{\bar{Z}}^{\bar{\jmath}|\mu} + \frac{1}{2}G_{\bar{\imath}\bar{\jmath}}\tilde{\bar{Z}}^{\bar{\imath}}_\mu \tilde{\bar{Z}}^{\bar{\jmath}|\mu} - \frac{\mathrm{i}}{2}g_{i\bar{\jmath}}\left(\overline{\lambda}^{iA}\gamma^\mu \nabla_\mu \lambda^{\bar{\jmath}}_A + \overline{\lambda}^{\bar{\jmath}}_A \gamma^\mu \nabla_\mu \lambda^{iA}\right)\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
\mathcal{L}_{Pauli} = &-\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\left(\mathcal{F}^{\Lambda}_{\mu\nu} + L^{\Lambda}\overline{\psi}^{A}_{[\mu}\psi^{B}_{\nu]}\epsilon_{AB} + \overline{L}^{\Lambda}\overline{\psi}_{A[\mu}\psi_{B|\nu]}\epsilon^{AB}\right)\times \\
&\Big[\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}\overline{\psi}^{A}_{[\rho}\psi^{B}_{\sigma]}\epsilon_{AB} + \overline{\mathcal{N}}_{\Lambda\Sigma}\overline{L}^{\Sigma}\overline{\psi}_{A[\rho}\psi_{B|\sigma]}\epsilon^{AB} + \\
&- \mathrm{i}\overline{\mathcal{N}}_{\Lambda\Sigma}f^{\Sigma}_{i}\overline{\psi}^{A}_{[\rho}\gamma_{\sigma]}\lambda^{iB}\epsilon_{AB} - \mathrm{i}\mathcal{N}_{\Lambda\Sigma}\overline{f}^{\Sigma}_{\overline{\imath}}\overline{\psi}_{A[\rho}\gamma_{\sigma]}\lambda^{\overline{\imath}}{}_{B}\epsilon^{AB} + \\
&- \frac{\mathrm{i}}{4}\left(X_{\Lambda ij}\overline{\lambda}^{iA}\gamma_{\rho\sigma}\lambda^{jB}\epsilon_{AB} + \overline{X}_{\Lambda\overline{\imath}\overline{\jmath}}\overline{\lambda}^{\overline{\imath}}_{A}\gamma_{\rho\sigma}\lambda^{\overline{\jmath}}_{B}\epsilon^{AB}\right)\Big] + \\
&+ g_{i\overline{\jmath}}\left(\nabla_{\mu}z^{i}\overline{\lambda}^{\overline{\jmath}}_{A}\gamma^{\mu\nu}\psi^{A}_{\nu} + \nabla_{\mu}\overline{z}^{\overline{\jmath}}\overline{\lambda}^{iA}\gamma^{\mu\nu}\psi_{A\nu}\right) + \\
&- Q^{M}\left(p_{Mi}\nabla_{\mu}z^{i} - \bar{p}_{M\overline{\imath}}\nabla_{\mu}\bar{z}^{\overline{\imath}}\right)\overline{\psi}^{A}_{\nu}\gamma_{\sigma}\psi_{A|\rho}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} + \\
&+ \frac{\mathrm{i}}{2}\left[(\nabla_{k}g_{i\overline{\jmath}} - Q^{M}p_{Mk}g_{i\overline{\jmath}})\nabla_{\mu}z^{k} - (\nabla_{\overline{k}}g_{i\overline{\jmath}} - Q^{M}\bar{p}_{M\overline{k}}g_{i\overline{\jmath}})\nabla_{\mu}\overline{z}^{\overline{k}}\right]\overline{\lambda}^{iA}\gamma^{\mu}\lambda^{\overline{\jmath}}_{A}
\end{aligned} \tag{4.12}
$$

$$
\mathcal{L}_{gauge} = \mathcal{L}_{mass} + \mathcal{L}_{CS} - V(z,\overline{z}) \tag{4.13}
$$

$$
\mathcal{L}_{mass} = \left[2S_{AB}\overline{\psi}^{A}_{\mu}\gamma^{\mu\nu}\psi^{B}_{\nu} + \mathrm{i}g_{i\overline{\jmath}}W^{iAB}\overline{\lambda}^{\overline{\jmath}}_{A}\gamma_{\mu}\psi^{\mu}_{B} + \mathcal{M}_{iAjB}\overline{\lambda}^{iA}\lambda^{jB}\right] + \text{ h.c.} \tag{4.14}
$$

$$
\begin{aligned}
\mathcal{L}_{CS} \;=\; & -m^{MN}\mathcal{F}_{M}\wedge\left(B_{N} + \frac{1}{2}d_{\Lambda\Sigma N}A^{\Lambda}A^{\Sigma}\right) + \\
& -m^{MN}d_{(\Lambda\Sigma)N}\left(\mathcal{F}^{\Lambda} - m^{\Lambda P}B_{P}\right)\wedge A_{M}\wedge\left(A^{\Sigma} + m^{\Sigma Q}A_{Q}\right) + \\
& +\frac{1}{3}(\mathcal{T}_{\Lambda})_{\Sigma\Gamma}A^{\Lambda}\wedge A^{\Sigma}\wedge\left(\mathcal{F}^{\Gamma} - m^{\Lambda M}B_{M} - \frac{1}{8}f_{\Delta\Omega}{}^{\Gamma}A^{\Delta}\wedge A^{\Omega}\right) + \\
& -\frac{1}{2}m^{NP}\left(d_{\Lambda\Sigma N}\hat{T}_{\Gamma M}{}^{N} - d_{[\Delta\Sigma]N}f_{\Gamma\Lambda}{}^{\Delta}\right)A^{\Lambda}\wedge A^{\Sigma}\wedge A^{\Gamma}\wedge A_{P}
\end{aligned} \tag{4.15}
$$

$$
V(z,\bar{z}) = g_{i\overline{\jmath}}\left(k^{i}_{\Lambda}\overline{L}^{\Lambda} - k^{iM}\overline{L}_{M}\right)\left(\overline{k}^{\overline{\jmath}}_{\Sigma}L^{\Sigma} - \overline{k}^{\overline{\jmath}N}L_{N}\right) \tag{4.16}
$$

$$
\begin{aligned}
\mathcal{L}_{4f} = &\frac{1}{2}\left(L^\Lambda \overline{\psi}^A_\mu \psi^B_\nu \epsilon_{AB} + \overline{L}^\Lambda \overline{\psi}_{A\mu}\psi_{B\nu}\epsilon^{AB}\right)\left(\mathcal{N}_{\Lambda\Sigma}L^\Sigma \overline{\psi}^C_\rho \psi^D_\sigma \epsilon_{CD} + \overline{\mathcal{N}}_{\Lambda\Sigma}\overline{L}^\Sigma \overline{\psi}_{C\rho}\psi_{D\sigma}\epsilon^{CD}\right)\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} +\\
&-\frac{1}{2}\left(f^\Lambda_i \overline{\psi}^A_\mu \gamma_\rho \lambda^{iB}\epsilon_{AB} + \bar{f}^\Lambda_{\bar\imath}\overline{\psi}_{A\mu}\gamma_\rho \lambda^{\bar\imath}_B \epsilon^{AB}\right)\left(\overline{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma_j \overline{\psi}^C_\nu \gamma_\sigma \lambda^{jD}\epsilon_{CD} + \mathcal{N}_{\Lambda\Sigma}\bar{f}^\Sigma_{\bar\jmath}\overline{\psi}_{C\nu}\gamma_\sigma \lambda^{\bar\jmath}_D \epsilon^{CD}\right)\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} +\\
&-\frac{\mathrm{i}}{2} g_{i\bar\jmath}\overline{\lambda}^{iA}\gamma_\rho \lambda^{\bar\jmath}_B \overline{\psi}_{A\mu}\gamma_\sigma \psi^B_\nu \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} +\\
&-\frac{1}{4}\left(f^\Lambda_i \overline{X}_{\Lambda\bar\jmath\bar k}\overline{\psi}^A_\mu \gamma_\nu \lambda^{iB}\epsilon_{AB}\overline{\lambda}^{\bar\jmath}_C \gamma^{\mu\nu}\lambda^{\bar k}_D \epsilon^{CD} - \bar{f}^\Lambda_{\bar\imath} X_{\Lambda jk}\overline{\psi}_{A\mu}\gamma_\nu \lambda^{\bar\imath}_B \epsilon^{AB}\overline{\lambda}^{jC}\gamma^{\mu\nu}\lambda^{kD}\epsilon_{CD}\right) +\\
&-\frac{1}{6}\left(C_{ijk}\overline{\lambda}^{iA}\gamma^\mu \psi^B_\mu \overline{\lambda}^{jC}\lambda^{kD}\epsilon_{AC}\epsilon_{BD} - \overline{C}_{\bar\imath\bar\jmath\bar k}\overline{\lambda}^{\bar\imath}_A \gamma_\mu \psi^\mu_B \overline{\lambda}^{\bar\jmath}_C \lambda^{\bar k}_D \epsilon^{AC}\epsilon^{BD}\right) +\\
+\frac{1}{12}\Big\{ &\frac{3\mathrm{i}}{16}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}\left(C_{jkn}C_{i\ell m} g^{m\bar m} g^{n\bar n}\bar{f}^\Lambda_{\bar m}\bar{f}^\Sigma_{\bar n}\overline{\lambda}^{jA}\gamma_{ab}\lambda^{kB}\,\overline{\lambda}^{iC}\gamma^{ab}\lambda^{\ell D}\,\epsilon_{AB}\epsilon_{CD} + h.c.\right) +\\
&-\mathrm{i}\Big[\left(\nabla_i C_{jkl} + 2Q^M p_{Mi}C_{jkl} + 3C^m{}_{ij}C_{k\ell m}\right)\overline{\lambda}^{iA}\lambda^{jC}\overline{\lambda}^{kB}\lambda^{\ell D}\epsilon_{AB}\epsilon_{CD} + h.c.\Big] +\\
&+3\big[R_{\bar\imath\bar\jmath k\bar\ell} - \frac{3}{2}g_{i\bar\jmath}g_{k\bar\ell} + g_{i\bar\ell}g_{k\bar\jmath} - \frac{1}{2}g_{\ell\bar\jmath}\nabla_{\bar\ell}C^\ell{}_{ik} - \frac{1}{2}g_{k\bar k}\nabla_i C^{\bar k}{}_{\bar\jmath\bar\ell}\big]\overline{\lambda}^{iA}\lambda^{kB}\overline{\lambda}^{\bar\jmath}_A \lambda^{\bar\ell}_B\Big\}
\end{aligned}
\tag{4.17}
$$

In writing the kinetic terms of the Lagrangian we have denoted with a tilde the supercovariant field strengths defined as:

$$
\begin{aligned}
\tilde{\mathcal{F}}^\Lambda_{\mu\nu} \equiv &\mathcal{F}^\Lambda_{\mu\nu} + L^\Lambda \overline{\psi}^A_{[\mu}\psi^B_{\nu]}\epsilon_{AB} + \overline{L}^\Lambda \overline{\psi}_{A[\mu}\psi_{B|\nu]}\epsilon^{AB} +\\
&-\mathrm{i} f^\Lambda_i \overline{\psi}^A_{[\mu}\gamma_{\nu]}\lambda^{iB}\epsilon_{AB} - \mathrm{i}\overline{f}^\Lambda_{\bar\imath}\overline{\psi}_{A[\mu}\gamma_{\nu]}\lambda^{\bar\imath}{}_B \epsilon^{AB}
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
\tilde{\mathcal{F}}_{M|\mu\nu} \equiv &\mathcal{F}_{M|\mu\nu} + L_M \overline{\psi}^A_{[\mu}\psi^B_{\nu]}\epsilon_{AB} + \overline{L}_M \overline{\psi}_{A[\mu}\psi_{B|\nu]}\epsilon^{AB} +\\
&-\mathrm{i} f_{Mi}\overline{\psi}^A_{[\mu}\gamma_{\nu]}\lambda^{iB}\epsilon_{AB} - \mathrm{i}\overline{f}_{M\bar\imath}\overline{\psi}_{A[\mu}\gamma_{\nu]}\lambda^{\bar\imath}{}_B \epsilon^{AB}
\end{aligned}
\tag{4.19}
$$

$$
\tilde{H}_{M|\mu\nu\rho} \equiv H_{M|\mu\nu\rho} + 2\mathrm{i}P_M \overline{\psi}^A_{[\mu}\gamma_\rho \psi^A_{\nu]} - p_{Mi}\overline{\psi}_{A[\mu}\gamma_{\nu\rho]}\lambda^{iA} - \bar{p}_{M\bar\imath}\overline{\psi}^A_{[\mu}\gamma_{\nu\rho]}\lambda^{\bar\imath}{}_A \tag{4.20}
$$

$$
\tilde{Z}^i_\mu \equiv \nabla_\mu z^i - \overline{\lambda}^{iA}\psi_{A\mu} \tag{4.21}
$$

$$
\tilde{\overline{Z}}^{\bar\imath}_\mu \equiv \nabla_\mu \bar{z}^{\bar\imath} - \overline{\lambda}^{\bar\imath}_A \psi^A_\mu \tag{4.22}
$$

where the ordinary bosonic field strengths are denoted by:

$$
\mathcal{F}^\Lambda_{\mu\nu} = \partial_{[\mu}A^\Lambda_{\nu]} + \frac{1}{2}f_{\Sigma\Gamma}{}^\Lambda A^\Sigma_{[\mu}A^\Gamma_{\nu]} + m^{\Lambda M}B_{M|\mu\nu} \tag{4.23}
$$

$$
\mathcal{F}_{M|\mu\nu} = \partial_{[\mu}A_{M|\nu]} + \hat{T}_{\Lambda M}{}^N A^\Lambda_{[\mu}A_{N|\nu]} \tag{4.24}
$$

$$
H_{M|\mu\nu\rho} = \partial_{[\mu}B_{M|\nu\rho]} + T_{\Lambda M}{}^N A^\Lambda_{[\mu}B_{N|\nu\rho]} \tag{4.25}
$$

$$
\nabla_\mu z^i = \partial_\mu z^i + k^i_\Lambda A^\Lambda_\mu - k^{iM}A_{M\mu}\,. \tag{4.26}
$$

The mass matrices are given by:

$$S_{AB} = 0 \tag{4.27}$$

$$W^{i[AB]} = \epsilon^{AB}\left(k^i_\Lambda \overline{L}^\Lambda - k^{iM}\overline{L}_M\right) \tag{4.28}$$

$$\mathcal{M}_{jAkB} = \left[g_{\bar{\imath}[j}\left(f^\Lambda_{k]}\,\overline{k}^{\bar{\imath}}_\Lambda - f_{M|k]}\,\overline{k}^{\bar{\imath}M}\right) + \frac{1}{2}g_{\bar{\imath}j}\left(L^\Lambda\nabla_k\overline{k}^{\bar{\imath}}_\Lambda - L_M\nabla_k\overline{k}^{\bar{\imath}M}\right)\right]\epsilon_{AB}. \tag{4.29}$$

Note that the absence of a gravitino mass matrix $S_{AB} = 0$ is due to the absence of hypermultiplets and/or Fayet-Iliopoulos terms in our setting.

4.2 Supersymmetry transformation laws

The supersymmetry transformation laws leaving the Lagrangian invariant (up to total derivatives) are obtained from the superspace curvatures obtained in our geometric approach.

For the fermion fields we find:

$$\begin{aligned}
\delta\psi_{A\mu} =& \mathcal{D}_\mu\epsilon_A + \epsilon_{AB}T^-_{\mu\nu}\gamma^\nu\epsilon^B + \frac{1}{2}\epsilon_A Q^M\left(p_{Mi}\nabla_\mu z^i - \bar{p}_{M\bar{\imath}}\nabla_\mu\bar{z}^{\bar{\imath}}\right) + \\
&+ \left(A_{A\mu}^{\ B} + \gamma_{\mu\nu}A'^{\ B\nu}_{A}\right)\epsilon_B + \\
&- \frac{1}{2}Q^M\left(p_{Mi}\overline{\epsilon}_B\lambda^{iB} - \bar{p}_{M\bar{\imath}}\overline{\epsilon}^B\lambda^{\bar{\imath}}_B\right)\psi_{A\mu} + \\
&- \frac{\mathrm{i}}{2}\left(Q_i\overline{\lambda}^{iB}\epsilon_B + Q_{\bar{\imath}}\overline{\lambda}^{\bar{\imath}}_B\epsilon^B\right)\psi_{A\mu}
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
\delta\lambda^{iA} =& \mathrm{i}\tilde{Z}^i_\mu\gamma^\mu\epsilon^A + \mathcal{G}^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\epsilon^{AB} + W^{iAB}\epsilon_B + \\
&+ \frac{1}{2}\left(-C^i{}_{jk}\overline{\lambda}^{jA}\lambda^{kB} + \mathrm{i}C^i{}_{\bar{\jmath}\bar{k}}\overline{\lambda}^{\bar{\jmath}}_C\lambda^{\bar{k}}_D\epsilon^{AC}\epsilon^{BD}\right)\epsilon_B + \\
&+ \frac{1}{2}\lambda^{iA}Q^M\left(p_{Mi}\overline{\lambda}^{iB}\epsilon_B - \bar{p}_{M\bar{\imath}}\overline{\lambda}^{\bar{\imath}}_B\epsilon^B\right) \\
&+ \frac{\mathrm{i}}{2}\left(Q_i\overline{\lambda}^{iB}\epsilon_B + Q_{\bar{\imath}}\overline{\lambda}^{\bar{\imath}}_B\epsilon^B\right)\lambda_{iA} - \Gamma^i{}_{jk}\overline{\lambda}^{kB}\epsilon_B\lambda^{jA},
\end{aligned} \tag{4.31}$$

while for the boson fields we find:

$$\delta V_\mu^a = - \mathrm{i}\overline{\psi}_{A\mu}\gamma^a\epsilon^A - \mathrm{i}\overline{\psi}^A_\mu\gamma^a\epsilon_A \tag{4.32}$$

$$\begin{aligned}\delta A_\mu^\Lambda =& - 2L^\Lambda\overline{\epsilon}^A\psi_\mu^B\epsilon_{AB} - 2\overline{L}^\Lambda\overline{\epsilon}_A\psi_{B\mu}\epsilon^{AB} + \\ &+ \mathrm{i}f_i^\Lambda\overline{\epsilon}^A\gamma_\mu\lambda^{iB}\epsilon_{AB} + \mathrm{i}\bar{f}_{\bar{\imath}}^\Lambda\overline{\epsilon}_A\gamma_\mu\lambda_B^{\bar{\imath}}\epsilon^{AB}\end{aligned} \tag{4.33}$$

$$\begin{aligned}\delta A_{M\mu} =& - 2L_M\overline{\epsilon}^A\psi_\mu^B\epsilon_{AB} - 2\overline{L}_M\overline{\epsilon}_A\psi_{B\mu}\epsilon^{AB} + \\ &+ \mathrm{i}f_{Mi}\overline{\epsilon}^A\gamma_\mu\lambda^{iB}\epsilon_{AB} + \mathrm{i}\bar{f}_{M\bar{\imath}}\overline{\epsilon}_A\gamma_\mu\lambda_B^{\bar{\imath}}\epsilon^{AB}\end{aligned} \tag{4.34}$$

$$\begin{aligned}\delta B_{M|\mu\nu} =& 2\mathrm{i}P_M\left(\overline{\psi}_{A[\mu}\gamma_{\nu]}\epsilon^A + \overline{\psi}^A_{[\mu}\gamma_{\nu]}\epsilon_A\right) + p_{Mi}\overline{\epsilon}_A\gamma_{\mu\nu}\lambda^{iA} + \bar{p}_{M\bar{\imath}}\overline{\epsilon}^A\gamma_{\mu\nu}\lambda^{\bar{\imath}}_A + \\ &- (d_{\Lambda\Sigma M}A^\Sigma_{[\mu} + \hat{T}_{\Lambda M}{}^N A_{N[\mu})\Big(\mathrm{i}f_i^\Lambda\overline{\epsilon}^A\gamma_{\nu]}\lambda^{iB}\epsilon_{AB} + \mathrm{i}\bar{f}_{\bar{\imath}}^\Lambda\overline{\epsilon}_A\gamma_{\nu]}\lambda_B^{\bar{\imath}}\epsilon^{AB} + \\ &- 2L^\Lambda\overline{\epsilon}^A\psi_{\nu]}^B\epsilon_{AB} - 2\overline{L}^\Lambda\overline{\epsilon}_A\psi_{B\nu]}\epsilon^{AB}\Big)\end{aligned} \tag{4.35}$$

$$\delta z^i = \overline{\epsilon}_A\lambda^{iA} \tag{4.36}$$

where:

$$\begin{aligned}h_a =& \frac{1}{2}Q^M(p_{Mi}Z^i_a - \bar{p}_{M\bar{\imath}}\overline{Z}^{\bar{\imath}}_a) \\ =& \frac{\mathrm{i}}{4}Q^M\tilde{H}_M{}^{bcd}\epsilon_{abcd}\end{aligned} \tag{4.37}$$

$$T^-_{ab} = \left(\mathcal{N} - \bar{\mathcal{N}}\right)_{\Lambda\Sigma} L^\Lambda\left(\tilde{\mathcal{F}}^{-\Sigma}_{ab} + \frac{1}{8}\left(\nabla_i + Q^M p_{Mi}\right) f_j^\Sigma\overline{\lambda}^{iA}\gamma_{ab}\lambda^{jB}\epsilon_{AB}\right) \tag{4.38}$$

$$\mathcal{G}^{-i}_{ab} = \frac{\mathrm{i}}{2}g^{i\bar{\jmath}}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\bar{f}^\Lambda_{\bar{\jmath}}\left(\tilde{\mathcal{F}}^{-\Sigma} + \frac{1}{8}\left(\nabla_j + Q^M p_{Mj}\right) f_k^\Sigma\overline{\lambda}^{jA}\gamma_{ab}\lambda^{kB}\epsilon_{AB}\right) \tag{4.39}$$

$$A_{\mu|A}{}^B = - \frac{\mathrm{i}}{4}g_{i\bar{\jmath}}\left(\overline{\lambda}^{\bar{\jmath}}_A\gamma_\mu\lambda^{iB} - \delta^B_A\overline{\lambda}^{\bar{\jmath}}_C\gamma_\mu\lambda^{iC}\right) \tag{4.40}$$

$$A'_{\mu|A}{}^B = \frac{\mathrm{i}}{4}\,g_{i\bar{\jmath}}\left(\overline{\lambda}^{\bar{\jmath}}_A\gamma_\mu\lambda^{iB} - \frac{1}{2}\delta^B_A\overline{\lambda}^{\bar{\jmath}}_C\gamma_\mu\lambda^{iC}\right) \tag{4.41}$$

5 Some comments on the structure of the theory

In this section we want to make some observations on the structure of the Lagrangian and its properties, taking into account the constraints found in

superspace. A complete list of the relations found on the σ-model and gauge structure is given in Appendix D.

First of all we note that the kinetic Lagrangian (4.11) does not contain an explicit propagation equation for the tensors $B_{M|\mu\nu}$. Indeed, it is expressed in the first order formalism, through the auxiliary field $Y_{M\mu}$, whose field equation gives

$$Y_{M\mu} = \epsilon_{\mu\nu\rho\sigma}\sqrt{-g}\tilde{H}_M{}^{\nu\rho\sigma} \,. \tag{5.1}$$

The dualization condition for the tensor field-strength in terms of the scalar 1-form degree of freedom $Y_M^{(1)}$, (2.1), is obtained by the variation of the kinetic plus Pauli Lagrangian with respect to $\tilde{H}_{M\mu\nu\rho}$

$$\frac{\delta}{\delta\tilde{H}_{M\mu\nu\rho}}(\mathcal{L}_k + \mathcal{L}_{Pauli}) = 0 \,,$$

giving the desired result

$$Y_{M\mu} = \epsilon_{\mu\nu\rho\sigma}\sqrt{-g}\tilde{H}_M{}^{\nu\rho\sigma} = -2\mathrm{i}\left(p_{Mi}\tilde{Z}^i_\mu - \bar{p}_{M\bar{\imath}}\bar{\tilde{Z}}^{\bar{\imath}}_\mu\right) \,. \tag{5.2}$$

Note however that the dualization is recovered in a simpler way if working in the geometric approach, since it is immediately found by solving the superspace Bianchi identity for the 3-form H_M or, equivalently, by the superspace field equations from the superspace Lagrangian (C.1), see eq.s (D.13), (D.14).

The equation of motion of the field $B_{M\mu\nu}$, taking into account the duality relation (5.2), gives:

$$\tilde{\mathcal{F}}^-_{M|ab} = \overline{\mathcal{N}}_{M\Lambda}\tilde{\mathcal{F}}^{-\Lambda}_{ab} = \overline{\mathcal{N}}_{MX}\tilde{\mathcal{F}}^{-X}_{ab} + \overline{\mathcal{N}}_{MN}\tilde{\mathcal{F}}^{-N}_{ab} \tag{5.3}$$

This relation, together with its complex conjugate for $\tilde{\mathcal{F}}^+_{M|ab}$, allows to eliminate the Hodge-dual field-strengths F^M in terms of the fields F_M contained in the tensor multiplets.

An important observation is the following. As already touched on in Section 2, the $(2n_V + n_T)$-dimensional scalar sector of the off-shell theory is formulated in terms of a $(2n_V + 2n_T)$-dimensional embedding manifold, $\mathcal{M}_{(emb)}$, that can be endowed with a complex structure. We note, as we will better see in the analysis of Appendix E, that the matrix $g_{i\bar{\jmath}}$ is the metric of $\mathcal{M}_{(emb)}$. Actually $g_{i\bar{\jmath}}$ does not satisfy the metric postulate. This is because of (D.59), found from the superspace constraints, that we rewrite here:

$$\nabla_k g_{i\bar{\jmath}} = -g_{\ell\bar{\jmath}}C^\ell{}_{ik} \,. \tag{5.4}$$

However, if we modify the connection

$$\Gamma^i{}_{jk} \to \mathring{\Gamma}^i{}_{jk} \equiv \Gamma^i{}_{jk} - C^i{}_{jk} \tag{5.5}$$

then the new covariant derivative $\mathring{\nabla}[\mathring{\Gamma}]$ of $g_{i\bar{\jmath}}$ is zero:

$$\mathring{\nabla}_k g_{i\bar{\jmath}} = \nabla_k g_{i\bar{\jmath}} + C^\ell{}_{ik}\, g_{\ell\bar{\jmath}} = 0\,. \tag{5.6}$$

Furthermore, as observed in Appendix D, all the functions defined on $\mathcal{M}_{(emb)}$ carry a weight $(p, -p)$ with respect to the connections

$$\mathrm{i}\mathcal{Q}_i + Q^M p_{Mi}\,; \qquad \mathrm{i}\mathcal{Q}_{\bar{\imath}} - Q^M \bar{p}_{M\bar{\imath}} \tag{5.7}$$

respectively, that is with respect to the connection 1-form

$$\mathrm{i}\mathring{\mathcal{Q}} \equiv \mathrm{i}\mathcal{Q} + Q^M(p_{Mi}\nabla z^i - \bar{p}_{M\bar{\imath}}\nabla \overline{z}^{\bar{\imath}}) = \mathrm{i}\mathcal{Q} + \frac{\mathrm{i}}{2} Q^M Y^{(1)}_M \tag{5.8}$$

This allows to extend the definition of the $\mathring{\nabla}$ connection also to objects with non vanishing weight. Indeed if we define on $\mathcal{M}_{(emb)}$ a total connection $\mathring{\Gamma} + i\mathring{\mathcal{Q}}$ as follows

$$\mathring{\Gamma}^i_{jk} + i\, p\, \mathring{\mathcal{Q}}_k \delta^i_j \;\; \equiv \;\; \Gamma^i{}_{jk} + \mathrm{i}\, p\, \mathcal{Q}_k \delta^i_j - C^i{}_{jk} + p\, Q^M p_{Mk} \delta^i_j\,, \tag{5.9}$$

then in terms of the covariant derivative $\mathcal{D}$ defined in Appendix D, one has:

$$\mathring{\nabla}_j \delta^i_k = \mathcal{D}_j \delta^i_k - C^i{}_{jk}\,. \tag{5.10}$$

One can easily verify that all the constraints in Appendix D involving covariant derivatives of L^Λ, f^Λ_i, when expressed in terms of $\mathring{\nabla}$ become exactly those defining Special Geometry. Note that we also find, from the analysis of the superspace constraints:

$$L_M \;\; = \;\; \mathcal{N}_{M\Lambda} L^\Lambda \tag{5.11}$$
$$f_{Mi} \;\; = \;\; \overline{\mathcal{N}}_{M\Lambda} f^\Lambda_i \tag{5.12}$$

which identifies them with the lower part of the symplectic vectors of Special Geometry M_Λ and $h_{\Lambda i}$.

Now we recall that, as is well known, Special Geometry can be characterized by the expression of its Kähler–Hodge bundle whose curvature is (in the ungauged case)

$$\mathring{K} = d\mathring{\mathcal{Q}} = d\,\mathcal{Q} - \mathrm{i} d \left[\, Q^M (p_{Mi} dz^i - \bar{p}_{M\bar{\imath}} d\overline{z}^{\bar{\imath}})\right] \tag{5.13}$$

and by the curvature of the manifold, namely

$$\mathring{R}_{i\bar{\jmath}k\bar{\ell}}(\mathring{\Gamma}+\mathrm{i}\mathring{\mathcal{Q}}) = g_{i\bar{\jmath}}g_{k\bar{\ell}} + g_{k\bar{\jmath}}g_{i\bar{\ell}} - C_{ikm}C^{m}{}_{\bar{\jmath}\bar{\ell}}\,, \tag{5.14}$$

together with the relations:

$$\mathring{\nabla}_{\bar{\ell}}C^{\bar{k}}{}_{ij} = 0 \tag{5.15}$$

$$\mathring{\nabla}_{[i}C^{\bar{k}}{}_{\ell]j} = 0\,. \tag{5.16}$$

Eq. (5.13) implies the set of relations:

$$K_{ij} \equiv \nabla_{[i}\mathcal{Q}_{j]} = \mathrm{i}\nabla_{[i}(Q^M p_{M|j]}) \tag{5.17}$$

$$K_{i\bar{\jmath}} \equiv \nabla_i\mathcal{Q}_{\bar{\jmath}} = \mathrm{i}g_{i\bar{\jmath}} - \mathrm{i}\left[\nabla_i(Q^M\bar{p}_{M\bar{\jmath}}) + \nabla_{\bar{\jmath}}(Q^M p_{Mi})\right] \tag{5.18}$$

The simplest way to obtain equation (5.14) is to perform the integrability of the equations $\mathring{\nabla}_i f^\Lambda_j = \mathrm{i}\bar{f}^\Lambda_{\bar{k}}C^{\bar{k}}{}_{ij}$ and $\mathring{\nabla}_{\bar{\imath}} f^\Lambda_j = g_{j\bar{\imath}}L^\Lambda$, together with their complex conjugates.

If the integrability conditions are implemented in equations (D.32) - (D.38) using the $\Gamma + \mathrm{i}\mathcal{Q}$ connection (corresponding to the covariant derivative ∇), we find that besides $R^i{}_{j\bar{k}l}$, also the Riemann tensors $R^i{}_{jkl}$, $R^i{}_{j\bar{k}\bar{\ell}}$, $R^{\bar{\imath}}{}_{jkl}$, $R^i{}_{jk\bar{\ell}}$ $R^{\bar{\imath}}{}_{j\bar{k}l}$ are now different from zero, in agreement with the fact that $\mathcal{M}_{(emb)}$ is not a Kähler manifold anymore. Indeed using (5.9) we have

$$\begin{aligned}\mathring{R}^i{}_j(\mathring{\Gamma}+\mathrm{i}\mathring{\mathcal{Q}}) &= R^i{}_j(\Gamma+i\mathcal{Q}) + \nabla(-C^i{}_j + \frac{\mathrm{i}}{2}Q^M Y^{(1)}_M\delta^i_j) + \\ &\quad +(-C^i{}_k + \frac{\mathrm{i}}{2}Q^M Y^{(1)}_M\delta^i_k)\wedge(-C^k{}_j + \frac{\mathrm{i}}{2}Q^M Y^{(1)}_M\delta^k_j)\end{aligned} \tag{5.19}$$

where $\Gamma^i{}_j = \Gamma^i{}_{jk}\nabla z^k$ and $C^i{}_j = C^i{}_{jk}\nabla z^k$ are (1,0)-forms while $Y^{(1)}_M \equiv -2\mathrm{i}(p_{Mi}\nabla z^i - \bar{p}_{M\bar{\imath}}\nabla\bar{z}^{\bar{\imath}})$ is the 1-form corresponding to the degrees of free-

dom dual to the 3-form H_M. We found indeed, using (5.13):

$$R^k{}_{j|i\overline{\ell}} = -2\delta^k_{(i}g_{j)\overline{\ell}} + C^k{}_{\overline{\ell}\overline{\jmath}}C^{\overline{\jmath}}{}_{ij} - \nabla_{\overline{\ell}}C^k{}_{ij} \tag{5.20}$$

$$R^k{}_{j|i\ell} = \left(\nabla_{[i}C^k{}_{\ell]j} - C^k{}_{m[i}C^m{}_{\ell]j}\right) = \mathring{\nabla}_{[i}\, C^k{}_{\ell]j} \tag{5.21}$$

$$R^k{}_{j|\overline{\imath}\overline{\ell}} = 0 \tag{5.22}$$

$$\begin{aligned} R^{\overline{k}}{}_{j|i\overline{\ell}} &= \mathrm{i}\left(\nabla_{\overline{\ell}}C^{\overline{k}}{}_{ij} - C^{\overline{k}}{}_{\overline{\ell}\overline{\jmath}}C^{\overline{\jmath}}{}_{ij} - 2Q^M\overline{p}_{M\overline{\ell}}C^{\overline{k}}{}_{ij}\right) \\ &= \mathrm{i}\mathring{\nabla}_{\overline{\ell}}C^{\overline{k}}{}_{ij} \end{aligned} \tag{5.23}$$

$$\begin{aligned} R^{\overline{k}}{}_{j|i\ell} &= -\mathrm{i}\left(\nabla_{[i}C^{\overline{k}}{}_{\ell]j} + C^k{}_{j[i}C^{\overline{k}}{}_{\ell]k} + 2Q^M p_{M[i}C^{\overline{k}}{}_{\ell]j}\right) \\ &= -\mathrm{i}\mathring{\nabla}_{[i}C^{\overline{k}}{}_{\ell]j} \end{aligned} \tag{5.24}$$

$$R^{\overline{k}}{}_{j|\overline{\imath}\overline{\ell}} = 0 \tag{5.25}$$

together with the constraints:

$$g_{\overline{k}[\ell}C^{\overline{k}}{}_{i]j} = 0 \tag{5.26}$$

$$\nabla_{[i}g_{j]\overline{k}} = 0 \tag{5.27}$$

both implying, recalling the relations in Appendix D, that C_{ijk} is completely symmetric. Note that we can collect the Riemann tensors above in the 2-form expressions (where $C^i{}_j$, $C^{\overline{\imath}}{}_j$ are $(1,0)$-forms):

$$R^i{}_j = -2\delta^i_{(j}g_{k)\overline{\ell}}\nabla z^k \wedge \nabla\overline{z}^{\overline{\ell}} + \nabla C^i{}_j - C^i{}_k \wedge C^k{}_j - C^i{}_{\overline{k}} \wedge C^{\overline{k}}{}_j \tag{5.28}$$

$$R^{\overline{\imath}}{}_j = -\mathrm{i}\nabla C^{\overline{\imath}}{}_j + \mathrm{i}C^{\overline{\imath}}{}_k \wedge C^k{}_j + \mathrm{i}C^{\overline{\imath}}{}_{\overline{k}} \wedge C^{\overline{k}}{}_j + Y^{(1)}_M \wedge C^{\overline{\imath}}{}_j \tag{5.29}$$

One can check that expressing the curvatures $R^i{}_j$, $R^{\overline{\imath}}{}_j$ of our embedding space in terms of $\mathring{R}$ one finds that all the Riemann tensors are precisely related as in (5.19) to the corresponding quantities $\mathring{R}$. In particular, $\mathring{R}^i{}_{j\overline{k}l}$ reduces to eq. (5.19), as was to be expected. Requiring the vanishing of the other components one recovers the known relations of Special Geometry (5.15), (5.16).

A further observation is that in our theory we have, besides the quantities analogous to those of Special Geometry, also the σ-model extra functions P_M, p_{Mi}, Q^M, $C^i{}_{jk}$, k^{iM}, corresponding to extra structures and couplings arising from the presence of the vector-tensor multiplets. From this point of view, we could think of our model as a particular gauging of $N=2$ supergravity coupled to vector multiplets, where new structures and couplings have

been introduced. In fact, if we let all these extra structures go to zero, together with the electric and magnetic gauge coupling constants, all encoded in the symplectic quantities $\mathcal{T}_\alpha$ defined in section 3.1, we would recover the standard, ungauged special geometry. This is in line with the approach of [12, 13], where the gauged theory coupled to antisymmetric tensor fields was realized as a deformation of the ungauged theory coupled to vector multiplets, in such a way that, setting the coupling constant to zero, the tensors get decoupled and the ungauged theory retrieved. Such approach is somewhat complementary to the construction presented here, where our starting point was the construction of an $N = 2$ theory describing vector multiplets coupled to a number of vector-tensor ones. Consistency of the theory then required the introduction of structures and extra fields which allowed us to define a smooth limit to an ungauged theory where the antisymmetric tensors disappear in favor of their dual scalar fields. In this limit $\mathcal{M}_{(emb)}$ becomes the special Kähler scalar manifold, with metric $g_{i\bar{\jmath}}$.

Therefore, although obtained from a different perspective, the theory presented here can be viewed as originating from a deformation of an ungauged $N = 2$ theory coupled to vector multiplets only. The global symmetry group of this model is described by the isometry group $\mathrm{Isom}(\mathcal{M}_{(emb)})$ of the scalar manifold, endowed with a two-fold action [29]: A non-linear action on the scalar fields z^i and a linear electric-magnetic duality action on the vector field strengths and their magnetic duals, i.e. on $\mathcal{F}^\alpha$. With reference to this ungauged theory, we could interpret the symplectic matrices $(\mathcal{T}_\alpha)_\beta{}^\gamma$ as the generators of the gauge algebra embedded in the isometry algebra of $\mathcal{M}_{(emb)}$, namely expressed as linear combinations of the generators t_n of $\mathrm{Isom}(\mathcal{M}_{(emb)})$ (with $n = 1, \cdots, dim(\mathrm{Isom}(\mathcal{M}_{(emb)})))$ through an *embedding tensor* $\theta_\alpha{}^n$:

$$\mathcal{T}_{\alpha\beta}{}^\gamma \equiv \theta_\alpha{}^n \, t_{n\beta}{}^\gamma \,, \tag{5.30}$$

where $t_{n\beta}{}^\gamma$ are the $2(1 + n_V + n_T) \times 2(1 + n_V + n_T)$ symplectic realization of the generators t_n as infinitesimal duality transformations on $\mathcal{F}^\alpha$. Closure of the gauge group in $\mathrm{Isom}(\mathcal{M}_{(emb)})$ is then guaranteed by eq. (3.42), which is a quadratic condition on $\theta_\alpha{}^n$. Gauge invariance of the action and the absence of anomalies further require the linear constraint (3.31) on $\theta_\alpha{}^n$.

A generic special Kähler manifold may have no isometries at all. Somewhat implicit in our construction is the presence in $\mathcal{M}_{(emb)}$ of at least a number of isometries t_M which are parametrized by the scalars dual to the antisymmetric tensor fields $B_{\mu\nu M}$.

As a final observation, we stress that all the relations obtained on the scalar sector are in fact relations on the geometry of the embedding manifold $\mathcal{M}_{(emb)}$ since they include all the coordinates, that is also the auxiliary degrees of freedom dual to the tensors. Therefore, in order to obtain the geometry of the true σ-model underlying our theory we have to solve all these relations in terms of the physical scalar-field coordinates $(z^r, \bar{z}^{\bar{r}}, P_M)$ only. This requires a deep understanding of the embedding properties and will be the object of future investigation [25].

A Embedded algebra

We are going to show that the $\mathcal{T}_\alpha$, subect to (3.6) - (3.11) together with (3.32) - (3.35), close the algebra in the symplectic represntation:

$$[\mathcal{T}_\alpha, \mathcal{T}_\beta] = -\mathcal{T}_{\alpha\beta}{}^\gamma \, \mathcal{T}_\gamma \,. \tag{A.1}$$

Eq. (A.1) corresponds to 3 matrix-relations:

i)

$$[\mathcal{T}_\Lambda, \mathcal{T}_\Sigma] = -\hat{f}_{\Lambda\Sigma}{}^\Gamma \, \mathcal{T}_\Gamma \,. \tag{A.2}$$

It requires the constraints:

$$\hat{f}_{\Lambda\Gamma}{}^\Delta \hat{f}_{\Sigma\Delta}{}^\Pi - \hat{f}_{\Sigma\Gamma}{}^\Delta \hat{f}_{\Lambda\Delta}{}^\Pi = -\hat{f}_{\Lambda\Sigma}{}^\Delta \hat{f}_{\Delta\Gamma}{}^\Pi \tag{A.3}$$

$$f_{\Lambda X}{}^Y (\mathcal{T}_\Sigma)_{YZ} - f_{\Sigma X}{}^Y (\mathcal{T}_\Lambda)_{YZ} + (X \leftrightarrow Z) = -\frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma (\mathcal{T}_\Gamma)_{XZ} \tag{A.4}$$

Eq. (A.3) is satisfied by the Free Differential Algebra, see (3.14), while (A.4) is the condition for the tensor $(\mathcal{T}_\Gamma)_{XZ}$ to lie in the Chevalley-Eilenberg cohomology of the basis of left-invariant 1-forms.

ii)

$$\left[\mathcal{T}_\Lambda, \mathcal{T}^P\right] = -\hat{T}_{\Lambda M}{}^P \, \mathcal{T}^M \,, \tag{A.5}$$

It corresponds to the constraints

$$\hat{f}_{\Lambda\Sigma}{}^\Gamma \hat{T}_{\Gamma M}{}^P \, m^{\Delta M} - \hat{f}_{\Lambda\Gamma}{}^\Delta \hat{T}_{\Sigma M}{}^P \, m^{\Gamma M} = -\hat{T}_{\Lambda M}{}^P \hat{T}_{\Sigma N}{}^M \, m^{\Delta N} \tag{A.6}$$

$$\hat{f}_{\Lambda\Sigma}{}^\Gamma d_{(\Gamma\Delta)N} m^{NP} + \hat{f}_{\Lambda\Delta}{}^\Gamma d_{(\Sigma\Gamma)N} m^{NP} = \hat{T}_{\Lambda M}{}^P d_{(\Sigma\Delta)N} m^{NM} \tag{A.7}$$

Eq. (A.6) is satisfied using (3.7), (3.9) and the definitions of $\hat{T}$ and $\hat{f}$, while eq. (A.6) requires use of (3.11) and (3.32)

iii)

$$\left[\mathcal{T}^P, \mathcal{T}^Q\right] = 0\,, \tag{A.8}$$

It contains the 2 relations:

$$\begin{aligned} \hat{T}_{\Lambda N}{}^P m^{\Sigma N}\,\hat{T}_{\Sigma M}{}^Q m^{\Gamma M} - (P \leftrightarrow Q) &= 0 \quad (A.9)\\ m^{MN}\,\hat{T}_{(\Lambda|N}{}^P\,\hat{T}_{(\Gamma)|M}{}^Q - (P \leftrightarrow Q) &= 0 \quad (A.10) \end{aligned}$$

Eq. (A.9) is satisfied because of (3.9) and the definition of $\hat{T}$, implying the relation $m^{\Sigma N}\,\hat{T}_{\Sigma M}{}^Q = 0$; eq. (A.10) requires $m^{MN} = m^{NM}$.

B Superspace Bianchi identities and superspace parametrization of the curvatures

The superspace Bianchi identities are:

$$D\mathcal{R}^a{}_b = 0 \tag{B.1}$$

$$DT^a + \mathcal{R}^a{}_b V^b - \mathrm{i}\overline{\psi}^A\gamma^a\rho_A - \mathrm{i}\overline{\psi}_A\gamma^a\rho^A = 0 \tag{B.2}$$

$$\nabla\rho_A + \frac{1}{4}\gamma_{ab}\mathcal{R}^{ab}\psi_A - \frac{\mathrm{i}}{2}K\psi_A = 0 \tag{B.3}$$

$$\nabla\rho^A + \frac{1}{4}\gamma_{ab}\mathcal{R}^{ab}\psi^A + \frac{\mathrm{i}}{2}K\psi^A = 0 \tag{B.4}$$

$$\begin{aligned} \nabla F^\Lambda =\ & \nabla L^\Lambda\overline{\psi}^A\psi^B\epsilon_{AB} + \nabla\overline{L}^\Lambda\overline{\psi}_A\psi_B\epsilon^{AB} - 2L^\Lambda\overline{\psi}^A\rho^B\epsilon_{AB} +\\ & - 2\overline{L}^\Lambda\overline{\psi}_A\rho_B\epsilon^{AB} + m^{\Lambda M}\left(H_M - 2\mathrm{i}P_M\overline{\psi}_A\gamma_a\psi^A V^a\right) \end{aligned} \tag{B.5}$$

$$\begin{aligned} \nabla F_M =\ & \nabla L_M\overline{\psi}^A\psi^B\epsilon_{AB} + \nabla\overline{L}_M\overline{\psi}_A\psi_B\epsilon^{AB} - 2L_M\overline{\psi}^A\rho^B\epsilon_{AB} +\\ & - 2\overline{L}_M\overline{\psi}_A\rho_B\epsilon^{AB} \end{aligned} \tag{B.6}$$

$$\begin{aligned} \nabla H_M =\ & 2\mathrm{i}\nabla P_M\overline{\psi}_A\gamma_a\psi^A V^a - 2\mathrm{i}P_M\left(\overline{\psi}_A\gamma_a\rho^A + \overline{\psi}^A\gamma_a\rho_A\right)V^a +\\ & + 2P_M\overline{\psi}_A\gamma_a\psi^A\overline{\psi}_B\gamma^a\psi^B +\\ & + \Big[d_{\Lambda\Sigma M}\left(F^\Sigma - L^\Sigma\overline{\psi}^A\psi^B\epsilon_{AB} - \overline{L}^\Sigma\overline{\psi}_A\psi_B\epsilon^{AB}\right) +\\ & \quad + \hat{T}_{\Lambda M}{}^N\left(F_N - L_N\overline{\psi}^A\psi^B\epsilon_{AB} - \overline{L}_N\overline{\psi}_A\psi_B\epsilon^{AB}\right)\Big]\cdot\\ & \cdot\left(F^\Lambda - L^\Lambda\overline{\psi}^C\psi^D\epsilon_{CD} - \overline{L}^\Lambda\overline{\psi}_C\psi_D\epsilon^{CD}\right) \end{aligned} \tag{B.7}$$

$$
\begin{aligned}
D^2 z^i &= k^i_\Lambda \left(F^\Lambda - L^\Lambda \overline{\psi}^A \psi^B \epsilon_{AB} - \overline{L}^\Lambda \overline{\psi}_A \psi_B \epsilon^{AB} \right) + \\
&\quad - k^{iM} \left(F_M - L_M \overline{\psi}^A \psi^B \epsilon_{AB} - \overline{L}_M \overline{\psi}_A \psi_B \epsilon^{AB} \right) && \text{(B.8)} \\
\nabla^2 \lambda^{iA} &= -\frac{1}{4} \gamma_{ab} \mathcal{R}^{ab} \lambda^{iA} - \frac{\mathrm{i}}{2} K \lambda^{iA} + R^i{}_j \lambda^{jA} && \text{(B.9)} \\
\nabla^2 \lambda^{\bar\imath}_A &= -\frac{1}{4} \gamma_{ab} \mathcal{R}^{ab} \lambda^{\bar\imath}_A + \frac{\mathrm{i}}{2} K \lambda^{\bar\imath}_A + R^{\bar\imath}{}_{\bar\jmath} \lambda^{\bar\jmath}_A && \text{(B.10)}
\end{aligned}
$$

where:

$$
K = \mathrm{d}\mathcal{Q} \tag{B.11}
$$

is the curvature associated to the potential $\mathcal{Q}$, see eq.s (5.17), (5.18).

The superspace parametrizations of the curvatures are:

$$
\begin{aligned}
\mathcal{T}^a &= 0 && \text{(B.12)} \\
\mathcal{R}_{ab} &= \tilde{\mathcal{R}}_{abcd} V^c V^d - \mathrm{i} \left(\overline{\psi}_A \gamma_a \rho^A{}_{bc} + \overline{\psi}^A \gamma_a \rho_{Abc} \right) V^c + \\
&\quad - \mathrm{i} T^-_{ab} \epsilon_{AB} \overline{\psi}^A \psi^B - \mathrm{i} T^+_{ab} \epsilon^{AB} \overline{\psi}_A \psi_B + 2 \epsilon_{abcd} A'_B{}^{A|c} \overline{\psi}_A \gamma^d \psi^B + \\
&\quad - S_{AB} \overline{\psi}^A \gamma_{ab} \psi^B - \overline{S}^{AB} \overline{\psi}_A \gamma_{ab} \psi_B && \text{(B.13)} \\
\rho_A &= \rho_{Aab} V^a V^b + \epsilon_{AB} T^-_{ab} \gamma^b \psi^B V^a + h_a \psi_A V^a + \mathrm{i} S_{AB} \gamma_a \psi^B V^a + \\
&\quad + \frac{1}{2} \psi_A Q^M \left(p_{Mi} \overline{\psi}_B \lambda^{iB} - \bar{p}_{M\bar\imath} \overline{\psi}^B \lambda^{\bar\imath}_B \right) + \\
&\quad + \left(A_A{}^B{}_a + \gamma_{ab} A'_A{}^B{}_b \right) \psi_B V^a && \text{(B.14)} \\
\rho^A &= \rho^A_{ab} V^a V^b + \epsilon^{AB} T^+_{ab} \gamma^b \psi_B V^a - h_a \psi^A V^a + \mathrm{i} \overline{S}^{AB} \gamma_a \psi_B V^a + \\
&\quad - \frac{1}{2} \psi^A Q^M \left(p_{Mi} \overline{\psi}_B \lambda^{iB} - \bar{p}_{M\bar\imath} \overline{\psi}^B \lambda^{\bar\imath}_B \right) + \\
&\quad - \left(A^A_B{}_a + \gamma_{ab} A'^A_B{}_b \right) \psi^B V^a && \text{(B.15)} \\
H_M &= \tilde{H}_{M|abc} V^a V^b V^c + p_{Mi} \overline{\psi}_A \gamma_{ab} \lambda^{iA} V^a V^b + \bar{p}_{M\bar\imath} \overline{\psi}^A \gamma_{ab} \lambda^{\bar\imath}{}_A V^a V^b && \text{(B.16)} \\
F^\Lambda &= \tilde{\mathcal{F}}^\Lambda_{ab} V^a V^b + \mathrm{i} f^\Lambda_i \overline{\psi}^A \gamma_a \lambda^{iB} \epsilon_{AB} V^a + \mathrm{i} \overline{f}^\Lambda_{\bar\imath} \overline{\psi}_A \gamma_a \lambda^{\bar\imath}{}_B \epsilon^{AB} V^a && \text{(B.17)} \\
F_M &= \tilde{\mathcal{F}}_{Mab} V^a V^b + \mathrm{i} f_{Mi} \overline{\psi}^A \gamma_a \lambda^{iB} \epsilon_{AB} V^a + \mathrm{i} \overline{f}_{M\bar\imath} \overline{\psi}_A \gamma_a \lambda^{\bar\imath}{}_B \epsilon^{AB} V^a && \text{(B.18)} \\
Dz^i &= D_a z^i V^a + \overline{\psi}_A \lambda^{iA} && \text{(B.19)} \\
Dz^{\bar\imath} &= D_a \overline{z}^{\bar\imath} V^a + \overline{\psi}^A \lambda^{\bar\imath}{}_A && \text{(B.20)}
\end{aligned}
$$

$$\nabla\lambda^{iA} = \tilde{\nabla}_a\lambda^{iA}V^a + \mathrm{i}D_a z^i\gamma^a\psi^A + G^{i-}_{ab}\gamma^{ab}\epsilon^{AB}\psi_B + W^{iAB}\psi_B +$$
$$+\frac{1}{2}\lambda^{iA}Q^M\left(p_{Mj}\overline{\psi}_B\lambda^{jB} - p_{M\bar{\jmath}}\overline{\psi}^B\lambda^{\bar{\jmath}}_B\right)+$$
$$+\frac{1}{2}\left[-C^i{}_{jk}\overline{\lambda}^{jA}\lambda^{kB} + \mathrm{i}C^i{}_{\bar{\jmath}\bar{k}}\overline{\lambda}^{\bar{\jmath}}_C\lambda^{\bar{k}}_D\epsilon^{AC}\epsilon^{BD}\right]\psi_B \quad \text{(B.21)}$$
$$\nabla\lambda^{\bar{\imath}}{}_A = \tilde{\nabla}_a\lambda^{\bar{\imath}}{}_A V^a + \mathrm{i}D_a\overline{z}^{\bar{\imath}}\gamma^a\psi_A + G^{\bar{\imath}+}_{ab}\gamma^{ab}\epsilon_{AB}\psi^B + \overline{W}^{\bar{\imath}}{}_{AB}\psi^B +$$
$$-\frac{1}{2}\lambda^{\bar{\imath}}_A Q^M\left(p_{Mj}\overline{\psi}_B\lambda^{jB} - p_{M\bar{\jmath}}\overline{\psi}^B\lambda^{\bar{\jmath}}_B\right)+$$
$$-\frac{1}{2}\left[\overline{C}^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}}\overline{\lambda}^{\bar{\jmath}}_A\lambda^{\bar{k}}_B + \mathrm{i}\overline{C}^{\bar{\imath}}{}_{jk}\overline{\lambda}^{jC}\lambda^{kD}\epsilon_{AC}\epsilon_{BD}\right]\psi^B \quad \text{(B.22)}$$

C The superspace Lagrangian

$$\begin{aligned}
\mathcal{L}_{kin} \;=\;& R^{ab}V^cV^d\epsilon_{abcd} - 4\left[\overline{\psi}^A\gamma_a\rho_A - \overline{\psi}_A\gamma_a\rho^A\right]V^a + 3\beta_2 g_{i\bar{\jmath}}\overline{\lambda}^{iA}\gamma_b\lambda^{\bar{\jmath}}_A\,T_a\,V^aV^b + \\
&+\mathrm{i}\left[\beta_3\left(\mathcal{N}_{\Lambda\Sigma}\tilde{\mathcal{F}}^{\Lambda+}_{ab} + \overline{\mathcal{N}}_{\Lambda\Sigma}\tilde{\mathcal{F}}^{\Lambda-}_{ab}\right)\right]\times \\
&\times\left[F^\Sigma - \mathrm{i}\left(f^\Sigma_i\overline{\lambda}^{iA}\gamma_c\psi^B\epsilon_{AB} + f^\Sigma_{\bar{\imath}}\overline{\lambda}^{\bar{\imath}}_A\gamma_c\psi_B\epsilon^{AB}\right)V^c\right]V^aV^b + \\
&-\frac{1}{24}\beta_3\left(\overline{\mathcal{N}}_{\Lambda\Sigma}\tilde{\mathcal{F}}^{\Lambda-}_{ab}\tilde{\mathcal{F}}^{\Lambda-ab} - \mathcal{N}_{\Lambda\Sigma}\tilde{\mathcal{F}}^{\Lambda+}_{ab}\tilde{\mathcal{F}}^{\Lambda+ab}\right)\Omega + \\
&+d_1\mathcal{M}^{MN}Y_{Na}\left[H_M - \left(p_{Mi}\overline{\psi}_A\gamma_{bc}\lambda^{iA} + \bar{p}_{M\bar{\imath}}\overline{\psi}^A\gamma_{bc}\lambda^{\bar{\imath}}_A\right)V^bV^c\right]V^a + \\
&-\frac{1}{48}d_1\mathcal{M}^{MN}Y_{Ma}Y^a_N\Omega + \\
&+\mathrm{i}d_2\mathcal{M}^{MN}\left[H_M - \left(p_{Mi}\overline{\psi}_A\gamma_{ab}\lambda^{iA} + \bar{p}_{M\bar{\imath}}\overline{\psi}^A\gamma_{ab}\lambda^{\bar{\imath}}_A\right)V^aV^b\right]\times \\
&\times\left[p_{Ni}\left(Dz^i - \overline{\lambda}^{iA}\psi_A\right) - \bar{p}_{N\bar{\imath}}\left(D\overline{z}^{\bar{\imath}} - \overline{\lambda}^{\bar{\imath}}_A\psi^A\right)\right] + \\
&+\beta_1 G_{i\bar{\jmath}}\left[\tilde{Z}^i_a\left(D\overline{z}^{\bar{\jmath}} - \overline{\lambda}^{\bar{\jmath}}_A\psi^A\right) + \tilde{\overline{Z}}^{\bar{\jmath}}_a\left(Dz^i - \overline{\lambda}^{iA}\psi_A\right)\right]V^bV^cV^d\epsilon_{abcd} \\
&+2\gamma_1\left[G_{ij}\tilde{Z}^i_a\left(Dz^j - \overline{\lambda}^{jA}\psi_A\right) + \overline{G}_{\bar{\imath}\bar{\jmath}}\tilde{\overline{Z}}^{\bar{\imath}}_a\left(D\overline{z}^{\bar{\jmath}} - \overline{\lambda}^{\bar{\jmath}}_A\psi^A\right)\right]V^bV^cV^d\epsilon_{abcd} \\
&-\frac{1}{4}\left[\beta_1 G_{i\bar{\jmath}}\tilde{Z}^i_a\tilde{\overline{Z}}^{\bar{\jmath}\ a} + \gamma_1\left(G_{ij}\tilde{Z}^i_a\tilde{Z}^{j\ a} + \overline{G}_{\bar{\imath}\bar{\jmath}}\tilde{\overline{Z}}^{\bar{\imath}}_a\tilde{\overline{Z}}^{\bar{\jmath}\ a}\right)\right]\Omega \\
&+\mathrm{i}\beta_2 g_{i\bar{\jmath}}\left(\overline{\lambda}^{iA}\gamma_a\nabla\lambda^{\bar{\jmath}}_A + \overline{\lambda}^{\bar{\jmath}}_A\gamma_a\nabla\lambda^{iA}\right)V^bV^cV^d\epsilon_{abcd} \qquad \text{(C.1)}
\end{aligned}$$

Note that we have written in (4.11) the kinetic terms for the bosonic fields in first order formalism, introducing the auxiliary fields $\tilde{H}_{Na}$, $\tilde{\mathcal{F}}^\Lambda_{ab}$ $\tilde{\mathcal{F}}_{M|ab}$, $\tilde{Z}^i_a$, $\tilde{\overline{Z}}^{\bar{\jmath}}_a$

which turn out to be identified on shell with the corresponding supercovariant field-strengths defined in (4.18) - (4.22).

$$\begin{aligned}
\mathcal{L}_{Pauli} &= -\mathrm{i}\beta_3 F^\Lambda \left(\mathcal{N}_{\Lambda\Sigma} L^\Sigma \overline{\psi}^A \psi^B \epsilon_{AB} + \overline{\mathcal{N}}_{\Lambda\Sigma} \overline{L}^\Sigma \overline{\psi}_A \psi_B \epsilon^{AB}\right) + \\
&\quad -\beta_3 F^\Lambda \left(\overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma \overline{\lambda}^{iA} \gamma_a \psi^B \epsilon_{AB} + \mathcal{N}_{\Lambda\Sigma} \overline{f}_{\bar{\imath}}^\Sigma \overline{\lambda}^{\bar{\imath}}_A \gamma_a \psi_B \epsilon^{AB}\right) V^a + \\
&\quad +\beta_7 F^\Lambda \left(X_{\Lambda ij} \overline{\lambda}^{iA} \gamma_{ab} \lambda^{jB} \epsilon_{AB} - \overline{X}_{\Lambda \bar{\imath}\bar{\jmath}} \overline{\lambda}^{\bar{\imath}}_A \gamma_{ab} \lambda^{\bar{\jmath}}_B \epsilon^{AB}\right) V^a V^b + \\
&\quad +\beta_8 g_{i\bar{\jmath}} \left(Dz^i \overline{\lambda}^{\bar{\jmath}}_A \gamma^{ab} \psi^A + D\overline{z}^{\bar{\jmath}} \overline{\lambda}^{iA} \gamma^{ab} \psi_A\right) V^c V^d \epsilon_{abcd} \\
&\quad -\mathrm{i}\beta_9 Q^M \left(p_{Mi} Dz^i - p_{M\bar{\imath}} D\overline{z}^{\bar{\imath}}\right) \overline{\psi}^A \gamma_a \psi_A V^a \\
&\quad + \left(K_{i\bar{\jmath}k} Dz^k - K_{i\bar{\jmath}\bar{k}} D\overline{z}^{\bar{k}}\right) \overline{\lambda}^{iA} \gamma^a \lambda^{\bar{\jmath}}_A \epsilon_{abcd} V^b V^c V^d
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
\mathcal{L}_{gauge} &= \mathrm{i}\delta_1 \left(S_{AB} \overline{\psi}^A \gamma_{ab} \psi^B + \overline{S}^{AB} \overline{\psi}_A \gamma_{ab} \psi_B\right) V^a V^b + \\
&\quad +\mathrm{i}\delta_2 g_{i\bar{\jmath}} \left(W^{iAB} \overline{\lambda}^{\bar{\jmath}}_A \gamma^a \psi_B + \overline{W}^{\bar{\jmath}}_{AB} \overline{\lambda}^{iA} \gamma^a \psi^B\right) V^b V^c V^d \epsilon_{abcd} + \\
&\quad +\delta_6 \left(\mathcal{M}_{iAjB} \overline{\lambda}^{iA} \lambda^{jB} + \mathcal{M}^{AB}_{\bar{\imath}\bar{\jmath}} \overline{\lambda}^{\bar{\imath}}_A \lambda^{\bar{\jmath}}_B\right) \Omega - \delta_7 V \Omega
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
\mathcal{L}_{CS} &= a^{MN} \mathcal{F}_M B_N + r_{\Lambda\Sigma}{}^M dA_M A^\Lambda A^\Sigma + \\
&\quad +dA^\Lambda \left(b_{\Lambda\Sigma\Gamma} A^\Sigma A^\Gamma + b_{\Lambda\Sigma}{}^M A^\Sigma A_M + b_\Lambda{}^{MN} A_M A_N\right) + \\
&\quad +A^\Lambda A^\Sigma \left(t_{\Lambda\Sigma\Gamma\Delta} A^\Gamma A^\Delta + t_{\Lambda\Sigma\Gamma}{}^M A^\Gamma A_M + t_{\Lambda\Sigma}{}^{MN} A_M A_N\right)
\end{aligned} \tag{C.4}$$

The expressions of the invariant tensors in $\mathcal{L}_{CS}$ are fixed requiring gauge invariance of the full Lagrangian, as discussed below in section D.1. This is also a necessary condition for the Lagrangian to be supersymmetric.

$$\begin{aligned}
\mathcal{L}_{4f} &= \frac{\mathrm{i}}{2}\beta_3 \left(L^\Lambda \overline{\psi}^A \psi^B \epsilon_{AB} + \overline{L}^\Lambda \overline{\psi}_A \psi_B \epsilon^{AB}\right) \left(\mathcal{N}_{\Lambda\Sigma} L^\Sigma \overline{\psi}^C \psi^D \epsilon_{CD} + \overline{\mathcal{N}}_{\Lambda\Sigma} \overline{L}^\Sigma \overline{\psi}_C \psi_D \epsilon^{CD}\right) + \\
&- \frac{\mathrm{i}}{2}\beta_3 \left(f_i^\Lambda \overline{\psi}^A \gamma_a \lambda^{iB} \epsilon_{AB} + \overline{f}_{\bar{\imath}}^\Lambda \overline{\psi}_A \gamma_a \lambda^{\bar{\imath}}_B \epsilon^{AB}\right) \left(\overline{\mathcal{N}}_{\Lambda\Sigma} f_j^\Sigma \overline{\psi}^C \gamma_b \lambda^{jD} \epsilon_{CD} + \mathcal{N}_{\Lambda\Sigma} \overline{f}_{\bar{\jmath}}^\Sigma \overline{\psi}_C \gamma_b \lambda^{\bar{\jmath}}_D \epsilon^{CD}\right) V^a V^b + \\
&+ \alpha_3 \left(f_i^\Lambda \overline{X}_{\Lambda\bar{\jmath}\bar{k}} \overline{\psi}^A \gamma_c \lambda^{iB} \epsilon_{AB} \overline{\lambda}^{\bar{\jmath}}_C \gamma_{ab} \lambda^{\bar{k}}_D \epsilon^{CD} - \overline{f}_{\bar{\imath}}^\Lambda X_{\Lambda jk} \overline{\psi}_A \gamma_c \lambda^{\bar{\imath}}_B \epsilon^{AB} \overline{\lambda}^{jC} \gamma_{ab} \lambda^{kD} \epsilon_{CD}\right) V^a V^b V^c + \\
&+ \alpha_4 g_{i\bar{\jmath}} \overline{\lambda}^{iA} \gamma_a \lambda^{\bar{\jmath}}_B \overline{\psi}_A \gamma_b \psi^B V^a V^b + \\
&+ \alpha_5 \left(C_{ijk} \overline{\lambda}^{iA} \gamma_a \psi^B \overline{\lambda}^{jC} \lambda^{kD} \epsilon_{AC} \epsilon_{BD} - \overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}} \overline{\lambda}^{\bar{\imath}}_A \gamma_a \psi_B \overline{\lambda}^{\bar{\jmath}}_C \lambda^{\bar{k}}_D \epsilon^{AC} \epsilon^{BD}\right) V^b V^c V^d \epsilon_{abcd} +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{72}\Big\{\frac{3\mathrm{i}}{16}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}\left(C_{jkn}C_{i\ell m}g^{m\overline{m}}g^{n\overline{n}}\bar{f}^{\Lambda}_{\overline{m}}\bar{f}^{\Sigma}_{\overline{n}}\overline{\lambda}^{jA}\gamma_{ab}\lambda^{kB}\,\overline{\lambda}^{iC}\gamma^{ab}\lambda^{\ell D}\,\epsilon_{AB}\epsilon_{CD}+h.c.\right)+\\
&\quad-\mathrm{i}\Big[\left(\nabla_i C_{jkl}+2Q^M p_{Mi}C_{jkl}-3C^m{}_{ij}C_{k\ell m}\right)\overline{\lambda}^{iA}\lambda^{jC}\overline{\lambda}^{kB}\lambda^{\ell D}\epsilon_{AB}\epsilon_{CD}+h.c.\Big]+\\
&\quad+3\big[R_{i\bar{\jmath}k\bar{\ell}}-\frac{3}{2}g_{i\bar{\jmath}}g_{k\bar{\ell}}+g_{i\bar{\ell}}g_{k\bar{\jmath}}-\frac{1}{2}g_{\ell\bar{\jmath}}\nabla_{\bar{\ell}}C^{\ell}{}_{ik}-\frac{1}{2}g_{k\bar{k}}\nabla_i C^{\bar{k}}{}_{\bar{\jmath}\bar{\ell}}\big]\overline{\lambda}^{iA}\lambda^{kB}\overline{\lambda}^{\bar{\jmath}}_{A}\lambda^{\bar{\ell}}_{B}\Big\}\,\Omega
\end{aligned}
\tag{C.5}
$$

where

$$\Omega\equiv V^aV^bV^cV^d\epsilon_{abcd}$$

and
$\beta_1=\frac{2}{3}$, $\beta_2=-\frac{1}{3}$, $\beta_3=4\mathrm{i}$, $\beta_5=4$, $\beta_6=-4$, $\beta_7=\frac{1}{2}$, $\beta_8=1$, $\beta_9=4\mathrm{i}$, $\gamma_1=\frac{1}{3}$, $d_1=-\frac{1}{2}$, $d_2=-1$, $\delta_1=-4$, $\delta_2=\frac{2}{3}$, $\delta_6=\frac{1}{6}$, $\alpha_3=\frac{\mathrm{i}}{2}$, $\alpha_4=2\mathrm{i}$, $\alpha_5=-\frac{1}{9}$.

D Constraints on the σ-model and gauging

- **relations on the gauging:**

$$k^{iM}=-2\mathrm{i}m^{MN}\bar{p}_{N\bar{\jmath}}g^{i\bar{\jmath}} \tag{D.1}$$

$$k^i_\Lambda m^{\Lambda M}=0 \tag{D.2}$$

$$k^i_\Lambda L^\Lambda=k^{iM}L_M \tag{D.3}$$

$$f^\Lambda_i\epsilon_{AB}W^{iAB}+\bar{f}^\Lambda_{\bar{\imath}}\epsilon^{AB}\overline{W}^{\bar{\imath}}_{AB}+4m^{\Lambda M}P_M=0 \tag{D.4}$$

$$f_{Mi}\epsilon_{AB}W^{iAB}+\bar{f}_{M\bar{\imath}}\epsilon^{AB}\overline{W}^{\bar{\imath}}_{AB}=0 \tag{D.5}$$

$$Q^M p_{Mi}W^{i[AB]}=Q^M\bar{p}_{M\bar{\imath}}\overline{W}^{\bar{\imath}}_{[AB]}=0 \tag{D.6}$$

$$
\begin{aligned}
&g_{i(\bar{\jmath}}\left(\bar{f}^\Lambda_{\bar{k})}k^i_\Lambda-\bar{f}_{M|\bar{k})}k^{iM}\right)-g_{i\bar{\jmath}}C^i{}_{\bar{k}\bar{\ell}}\left(L^\Lambda\overline{k}^{\bar{\ell}}_\Lambda-L_M\overline{k}^{\bar{\ell}M}\right)=\\
&=\frac{1}{6}g_{i\bar{\jmath}}\left(\nabla_{\bar{k}}k^i_\Lambda\overline{L}^\Lambda-\nabla_{\bar{k}}k^{iM}\overline{L}_M\right)
\end{aligned}
\tag{D.7}
$$

$$\nabla_{(i}\left(g_{j)\bar{k}}k^{\bar{k}}_\Lambda\right)=-C^k{}_{ij}g_{k\bar{k}}\,k^{\bar{k}}_\Lambda \tag{D.8}$$

$$\nabla_{(i}\left(g_{j)\bar{k}}k^{\bar{k}M}\right)=-C^k{}_{ij}g_{k\bar{k}}\,k^{\bar{k}M} \tag{D.9}$$

from which we deduce, using (D.46) and (D.48)

$$\overline{L}^\Lambda\hat{T}_{\Lambda M}{}^N P_N Q^M=0 \tag{D.10}$$

The above relation is automatically satisfied if $Q^M \propto m^{MN} P_N$.

$$m^{MN} P_M = -\frac{\mathrm{i}}{2} Q^M \left(p_{Mi} k^{iN} - \bar{p}_{M\bar{\imath}} k^{\bar{\imath}N} \right) \tag{D.11}$$

$$\begin{aligned}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} f_i^\Sigma (k_\Lambda^i \overline{L}^\Lambda - k^{iM} \overline{L}_M) = \\ = \mathrm{i} Q^M \left[p_{Mi} (k_\Lambda^i + \mathcal{N}_{\Lambda N} k^{iN}) - \bar{p}_{M\bar{\imath}} (k_\Lambda^{\bar{\imath}} + \mathcal{N}_{\Lambda N} k^{\bar{\imath}N}) \right]\end{aligned} \tag{D.12}$$

- **relations on the gauge structure:**

$$3\mathrm{i} \tilde{H}_M{}^{abc} = \epsilon^{abcd} \left(p_{Mi} Z_d^i - \bar{p}_{M\bar{\imath}} \overline{Z}_d^{\bar{\imath}} \right) \tag{D.13}$$

$$Y_{Ma} = \epsilon_{abcd} \tilde{H}_M^{bcd} \tag{D.14}$$

$$\tilde{\mathcal{F}}_{M|ab}^- = \overline{\mathcal{N}}_{\Lambda M} \tilde{\mathcal{F}}_{ab}^{-\Lambda} \tag{D.15}$$

together with:

$$\mathcal{N}_{\Lambda M} L^\Lambda = L_M \tag{D.16}$$

$$\overline{\mathcal{N}}_{\Lambda M} f_i^\Lambda = f_{Mi}\,. \tag{D.17}$$

- **relations on the σ-model:**

$$\left(\mathcal{N} - \bar{\mathcal{N}} \right)_{\Lambda\Sigma} L^\Lambda \overline{L}^\Sigma = -\mathrm{i} \tag{D.18}$$

$$\left(\mathcal{N} - \bar{\mathcal{N}} \right)_{\Lambda\Sigma} f_i^\Lambda f_{\bar{\jmath}}^\Sigma = -\mathrm{i}\, g_{i\bar{\jmath}} \tag{D.19}$$

$$\left(\mathcal{N} - \bar{\mathcal{N}} \right)_{\Lambda\Sigma} L^\Lambda f_i^\Sigma = 0 \tag{D.20}$$

$$\left(\mathcal{N} - \overline{\mathcal{N}} \right)^{-1|\Lambda\Sigma} = \mathrm{i} \left(\bar{L}^\Lambda L^\Sigma + g^{i\bar{\jmath}} f_i^\Lambda f_{\bar{\jmath}}^\Sigma \right) \tag{D.21}$$

We also find:

$$\nabla_{\bar{\jmath}} \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma = g_{i\bar{\jmath}} (\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} L^\Sigma \tag{D.22}$$

$$\nabla_{(i} \overline{\mathcal{N}}_{\Lambda\Sigma} f_{j)}^\Sigma = \mathrm{i} (\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} f_{\bar{k}}^\Sigma C^{\bar{k}}{}_{ij} \tag{D.23}$$

Furthermore:

$$\left(\nabla_i + Q^M p_{Mi}\right) L^\Lambda = f_i^\Lambda \tag{D.24}$$
$$\left(\nabla_i - Q^M p_{Mi}\right) \overline{L}^\Lambda = 0 \tag{D.25}$$
$$\left(\nabla_{\bar\imath} - Q^M \bar{p}_{M\bar\imath}\right) L^\Lambda = 0 \tag{D.26}$$
$$\left(\nabla_{\bar\imath} + Q^M \bar{p}_{M\bar\imath}\right) \overline{L}^\Lambda = \bar{f}_{\bar\imath}^\Lambda \tag{D.27}$$
$$\left(\nabla_i + Q^M p_{Mi}\right) L_M = f_{Mi} \tag{D.28}$$
$$\left(\nabla_i - Q^M p_{Mi}\right) \overline{L}_M = 0 \tag{D.29}$$
$$\left(\nabla_{\bar\imath} - Q^M \bar{p}_{M\bar\imath}\right) L_M = 0 \tag{D.30}$$
$$\left(\nabla_{\bar\imath} + Q^M \bar{p}_{M\bar\imath}\right) \overline{L}_M = \bar{f}_{M\bar\imath} \tag{D.31}$$

$$\left(\nabla_{\bar\jmath} - Q^M \bar{p}_{M\bar\imath}\right) f_i^\Lambda = L^\Lambda g_{i\bar\jmath} \tag{D.32}$$
$$\left(\nabla_{\bar\jmath} - Q^M \bar{p}_{M\bar\imath}\right) f_{Mi} = L_M g_{i\bar\jmath} \tag{D.33}$$

$$\left(\nabla_{[i} + Q^M p_{M[i}\right) f_{j]}^\Lambda = 0 \tag{D.34}$$
$$\left(\nabla_{[i} + Q^M p_{M[i}\right) f_{M|j]} = 0 \tag{D.35}$$
$$\left(\nabla_i + Q^M p_{Mi}\right) f_j^\Lambda = -f_k^\Lambda C^k{}_{ij} + \mathrm{i}\bar{f}_{\bar k}^\Lambda C^{\bar k}{}_{ij} \tag{D.36}$$
$$= 8 f_k^\Lambda G^k{}_{ij} + 8\mathrm{i}\overline{L}^\Lambda T_{ij} \tag{D.37}$$
$$\left(\nabla_i + Q^M p_{Mi}\right) f_{Mj} = -f_{Mk} C^k{}_{ij} + \mathrm{i}\bar{f}_{M\bar k} C^{\bar k}{}_{ij} \tag{D.38}$$
$$= 8 f_{Mk} G^k{}_{ij} + 8\mathrm{i}\overline{L}_M T_{ij} \tag{D.39}$$
$$C^i{}_{[jk]} = C^i{}_{[\bar\jmath\bar k]} = 0 \tag{D.40}$$

where

$$T_{ij} = \frac{1}{8}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} L^\Lambda \left(\nabla_i + Q^M p_{Mi}\right) f_j^\Sigma \tag{D.41}$$
$$\mathcal{G}^k{}_{ij} = \frac{\mathrm{i}}{8} g^{k\bar k}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} \bar{f}_{\bar k}^\Lambda \left(\nabla_i + Q^M p_{Mi}\right) f_j^\Sigma \tag{D.42}$$

In the relations above, the covariant derivative ∇ is covariant with respect to the Levi-Civita connection on the embedding manifold and to the Kähler connection $\mathcal{Q}$, under which $L^\Lambda, L_M, f_i^\Lambda, f_{Mi}$ have weight $+1$, their complex conjugates carrying weight -1. However, let us observe

that all the above relations can be written in terms of a new covariant derivative which includes also a connection $Q^M(p_{Mi}\nabla z^i - \bar{p}_{M\bar{\imath}}\nabla \overline{z}^{\bar{\imath}})$:

$$\mathcal{D}_i \equiv \nabla_i + p\, Q^M p_{Mi} \tag{D.43}$$
$$\mathcal{D}_{\bar{\imath}} \equiv \nabla_{\bar{\imath}} - p\, Q^M \bar{p}_{M\bar{\imath}} \tag{D.44}$$

if we assume that the sections $L^\Lambda, L_M, f_i^\Lambda, f_{Mi}$ all carry the same weight $+1$ with respect to the Kähler connection (their complex conjugates carrying the opposite weight).

$$P_M = -\frac{1}{2}\left[2d_{(\Lambda\Sigma)M}L^\Lambda\overline{L}^\Sigma + \hat{T}_{\Lambda M}{}^N\left(\overline{L}^\Lambda L_N + L^\Lambda\overline{L}_N\right)\right] \tag{D.45}$$
$$\begin{aligned}\nabla_i P_M &= p_{Mi}\\ &= -\frac{1}{2}\left[2d_{(\Lambda\Sigma)M}\overline{L}^\Lambda f_i^\Sigma + \hat{T}_{\Lambda M}{}^N\left(\overline{L}^\Lambda f_{iN} + f_i^\Lambda\overline{L}_N\right)\right]\end{aligned} \tag{D.46}$$

together with the constraints

$$d_{(\Lambda\Sigma)M}L^\Lambda L^\Sigma + \hat{T}_{\Lambda M}{}^N L^\Lambda L_N = 0\,, \tag{D.47}$$
$$2d_{(\Lambda\Sigma)M}L^\Lambda f_i^\Sigma + \hat{T}_{\Lambda M}{}^N\left(L^\Lambda f_{iN} + f_i^\Lambda L_N\right) = 0 \tag{D.48}$$

$$\nabla_{[i}p_{Mj]} = 0 \tag{D.49}$$
$$\nabla_i p_{Mj} = -p_{Mk}C^k{}_{ij} \tag{D.50}$$
$$\nabla_i p_{Mj} = 8p_{Mk}G^k{}_{ij} \tag{D.51}$$
$$P_M\overline{T}_{\bar{\imath}\bar{\jmath}} = -\frac{1}{8}p_{Mk}C^k{}_{\bar{\imath}\bar{\jmath}} \tag{D.52}$$
$$= -\frac{\mathrm{i}}{8}\left[d_{(\Lambda\Sigma)M}\bar{f}_{\bar{\imath}}^\Lambda\bar{f}_{\bar{\jmath}}^\Sigma + \hat{T}_{\Lambda M}{}^N\bar{f}_{(\bar{\imath}}^\Lambda\bar{f}_{N|\bar{\jmath})}\right] \tag{D.53}$$

$$\nabla_i\bar{p}_{M\bar{\jmath}} = \nabla_{\bar{\jmath}}p_{Mi} \tag{D.54}$$
$$\begin{aligned}\nabla_i\nabla_{\bar{\jmath}}P_M &= \frac{1}{2}\left[2d_{(\Lambda\Sigma)M}f_i^\Lambda\bar{f}_{\bar{\jmath}}^\Sigma + \hat{T}_{\Lambda M}{}^N\left(f_i^\Lambda\bar{f}_{M\bar{\jmath}} + \bar{f}_{\bar{\jmath}}^\Lambda f_{Ni}\right)\right]\\ &= \frac{1}{2}g_{i\bar{\jmath}}P_M\end{aligned} \tag{D.55}$$

$$\mathcal{M}^{MN} p_{Mi} p_{Nj} = 2G_{ij} \tag{D.56}$$
$$\mathcal{M}^{MN} p_{Mi} \bar{p}_{N\bar{\jmath}} = -2G_{i\bar{\jmath}} + 2g_{i\bar{\jmath}} \tag{D.57}$$

$$-\frac{1}{3}\left(\nabla_k g_{i\bar{\jmath}} - Q^M p_{Mk} g_{i\bar{\jmath}}\right) = \mathrm{i} K_{i\bar{\jmath}k} \tag{D.58}$$
$$\nabla_k g_{i\bar{\jmath}} = -g_{\ell\bar{\jmath}} C^{\ell}{}_{ik} \tag{D.59}$$
$$X_{\Lambda ij} = \mathrm{i}(\mathcal{N} - \bar{\mathcal{N}})_{\Lambda\Sigma} \bar{f}^{\Sigma}_{\bar{\ell}} \bar{g}^{k\bar{\ell}} g_{i\bar{k}} C^{\bar{k}}{}_{jk} \tag{D.60}$$

$$g_{i\bar{k}} C^{\bar{k}}{}_{(jk)} = \frac{1}{3}\left(C_{ijk} + C_{jki} + C_{kij}\right) \tag{D.61}$$
$$= f^{\Lambda}_{k} X_{\Lambda(ij)} \tag{D.62}$$
$$C_{ijk} = C_{kji} \tag{D.63}$$
$$8\mathrm{i} g_{\ell\bar{\imath}} \mathcal{G}^{\ell}{}_{jk} + g_{\ell\bar{\imath}} C^{\ell}{}_{jk} + \bar{f}^{\Lambda}_{\bar{\imath}} X_{\Lambda(jk)} = 0 \tag{D.64}$$
$$g_{k\bar{k}} \nabla_{\bar{\ell}} C^{k}{}_{ij} = g_{j\bar{\jmath}} \nabla_i C^{\bar{\jmath}}{}_{\overline{k\ell}} \tag{D.65}$$

From the above relations we find

$$\hat{T}_{\Lambda M}{}^{N} (\mathcal{N} - \overline{\mathcal{N}})_{N\Sigma} L^{\Lambda} f^{\Sigma}_{k} C^{k}{}_{\overline{\imath\jmath}} = 0 \tag{D.66}$$
$$g_{k\bar{\jmath}} C^{\bar{\jmath}}{}_{j\ell} g^{\ell\bar{\imath}} = C^{\bar{\imath}}{}_{jk} \tag{D.67}$$

D.1 Constraints from gauge invariance

The Lagrangian must be gauge-invariant up to total derivatives. In checking this property, in particular two different sectors in the gauge variation of the Lagrangian do not depend on scalar fields and hence should cancel out each other: the first comes from $\mathcal{L}_{CS}$ and the second from the topologial sector of $\mathcal{L}_{Kin}$:

$$\mathcal{L}_{Kin,top} \propto \mathrm{Re}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda} F^{\Sigma}$$

which contributes non trivially when we consider gaugings with a non trivial action on the kinetic matrix of the gauge fields, that is gaugings with a non block-diagonal symplectic embedding. To have gauge invariance of $\mathcal{L}_{CS} + \mathcal{L}_{kin}$, we must take into account the possible constant contributions from the gauge variation of the kinetic Lagrangian [27].

The gauge transformations of the fields under vector-gauge transformations with parameter ϵ^Λ, ϵ_M, together with the symplectic embedding and the gauge transformation of $\mathcal{N}$, are given in Section 3.1.

We found the following set of conditions on the constant couplings:

$$\begin{aligned}
a^{MN} &= 4m^{MN} && \text{(D.68)}\\
b_{\Lambda[\Sigma\Gamma]} &= -\frac{2}{3}\left[(\mathcal{T}_\Gamma)_{\Lambda\Sigma} - (\mathcal{T}_\Sigma)_{\Lambda\Gamma}\right] && \text{(D.69)}\\
b_{(\Lambda\Sigma)}{}^M &= -4m^{MN}d_{(\Lambda\Sigma)N} && \text{(D.70)}\\
b_\Lambda{}^{[MN]} &= 2\,\hat{T}_{\Lambda P}{}^N m^{MP} && \text{(D.71)}\\
r_{[\Lambda\Sigma]}{}^M &= 2\,m^{MN}d_{[\Lambda\Sigma]N} && \text{(D.72)}\\
t_{[\Lambda\Sigma\Gamma\Delta]} &= \frac{1}{2}f_{[\Lambda\Sigma}{}^\Omega(\mathcal{T}_\Gamma)_{\Delta]\Omega} && \text{(D.73)}\\
t_{[\Lambda\Sigma\Gamma]}{}^M &= -2m^{MN}d_{\Delta[\Lambda|N}f_{\Sigma\Gamma]}{}^\Delta && \text{(D.74)}\\
t_{[\Lambda\Sigma]}{}^{[MN]} &= \hat{T}_{[\Gamma|P}{}^N m^{MP}f_{\Lambda\Sigma}{}^\Gamma && \text{(D.75)}
\end{aligned}$$

E The vector-tensor σ-model metric

Let us start from the (ungauged) kinetic term of special geometry:

$$\mathcal{L}_{kin} = g_{i\bar{\jmath}}\partial_\mu z^i\partial^\mu\overline{z}^{\bar{\jmath}} \tag{E.1}$$

In light of eq. (D.13), we want to dualize

$$-2\mathrm{i}\left(p_{Mi}dz^i - \bar{p}_{M\bar{\imath}}d\overline{z}^{\bar{\imath}}\right) \equiv Y_M^{(1)} \tag{E.2}$$

To this aim, let us assume the following Ansatz for the metric:

$$g_{i\bar{\jmath}} = \hat{G}_{i\bar{\jmath}} + \mathcal{M}^{MN}p_{Mi}\bar{p}_{N\bar{\jmath}} \tag{E.3}$$

and let us decompose $p_{Mi}dz^i$ into real and imaginary parts:

$$\begin{aligned}
p_{Mi}dz^i &= \frac{1}{2}\left(p_{Mi}dz^i + \bar{p}_{M\bar{\imath}}d\overline{z}^{\bar{\imath}}\right) + \frac{\mathrm{i}}{4}Y_M^{(1)}\\
&= \frac{1}{2}\nabla P_M + \frac{\mathrm{i}}{4}Y_M^{(1)}
\end{aligned} \tag{E.4}$$

In terms of the new variables, the Lagrangian (E.1) reads:

$$\mathcal{L}_{kin} = \frac{1}{2}G_{ij}\partial_\mu z^i\partial^\mu z^j + G_{i\bar{\jmath}}\partial_\mu z^i\partial^\mu\overline{z}^{\bar{\jmath}} + \frac{1}{2}G_{\bar{\imath}\bar{\jmath}}\partial_\mu\overline{z}^{\bar{\imath}}\partial^\mu\overline{z}^{\bar{\jmath}} + \frac{1}{16}\mathcal{M}^{MN}Y_{M\mu}Y_N{}^\mu \tag{E.5}$$

where

$$
\begin{aligned}
G_{ij} &= \frac{1}{2}\mathcal{M}^{MN}p_{Mi}p_{Nj} && \text{(E.6)}\\
G_{\bar{\imath}\bar{\jmath}} &= (G_{ij})^* && \text{(E.7)}\\
G_{i\bar{\jmath}} &= g_{i\bar{\jmath}} - \frac{1}{2}\mathcal{M}^{MN}p_{Mi}\bar{p}_{N\bar{\jmath}} && \text{(E.8)}
\end{aligned}
$$

To perform the dualization on the vector-tensor multiplet sector, we introduce the Lagrange multiplier $Y_{M\mu}$ and add to the Lagrangian (E.5) the term

$$-\frac{1}{8}\mathcal{M}^{MN}Y_{M\mu}H_{M\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \tag{E.9}$$

Varying then the new Lagrangian with respect to $Y_{M\mu}$ we obtain

$$Y_M{}^{\mu} = H_{M\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \tag{E.10}$$

We observe that we are using a redundant parametrization of the scalar manifold in terms of the $2n_V+2n_T$ coordinates $z^i, \overline{z}^{\bar{\imath}}$. Actually the scalar manifold has real dimension $2n_V + n_T$ and we want parametrize it with general coordinates ϕ_α. In order to know the constraints on the geometry coming from supersymmetry, it is then necessary to pull back all the differentials appearing in the relations found from the Bianchi Identities and the Lagrangian in terms of the differentials $\mathrm{d}\phi^\alpha$. The simplest way to perform the pull-back is to choose a particular system of coordinates: $\phi^\alpha = \{z^r, \overline{z}^{\bar{r}}, P_M\}$ so that

$$
\begin{aligned}
&\partial_\alpha z^i = \{\delta^i_r, 0, \partial^M z^i \equiv \xi^{Mi}\} && \text{(E.11)}\\
&\partial_\alpha \overline{z}^{\bar{\imath}} = \{0, \delta^{\bar{r}}_\alpha, \partial^M \overline{z}^{\bar{\imath}} \equiv \overline{\xi}^{M\bar{\imath}}\} && \text{(E.12)}\\
&\partial_\alpha P_M = p_{Mi}\partial_\alpha z^i + \bar{p}_{M\bar{\imath}}\partial_\alpha \overline{z}^{\bar{\imath}} = \{p_{Mr}, \bar{p}_{M\bar{r}}, \delta_{MN}\} && \text{(E.13)}
\end{aligned}
$$

$$G_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta} + \frac{1}{2}\mathcal{M}^{MN}p_{M\alpha}p_{N\beta} & \mathcal{M}^{MN}p_{N\beta} \\ \mathcal{M}^{NR}p_{R\alpha} & \mathcal{M}^{MN} \end{pmatrix} \tag{E.14}$$

A detailed analysis of the σ-model geometry will be presented in [25].

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